Logic through the lens of structural graph theory

Michał Pilipczuk

University of Warsaw

Finite and Algorithmic Model Theory

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(Ob)structural graph theory

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Structure

Decompositions exposing various properties:

— **trees** or **covers**,

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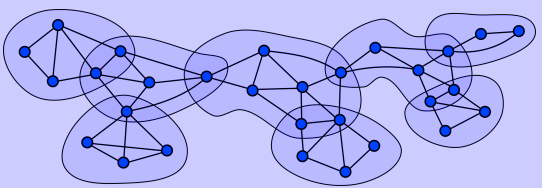

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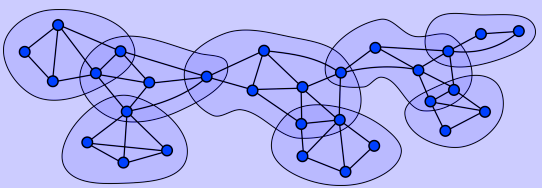

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Object embedded in the considered graph:

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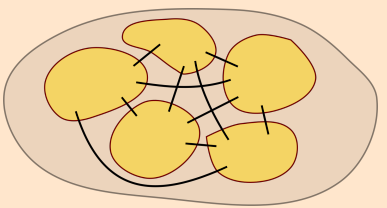

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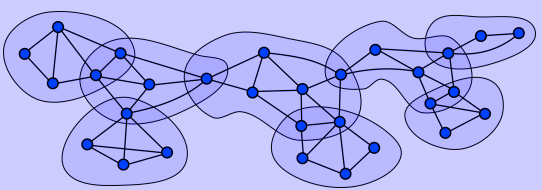

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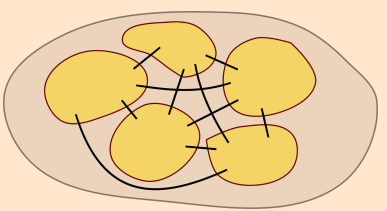

Theorem Template

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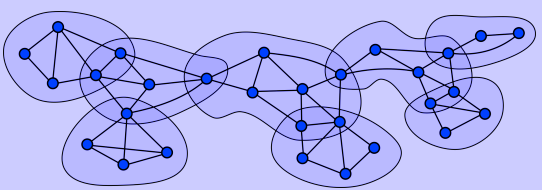

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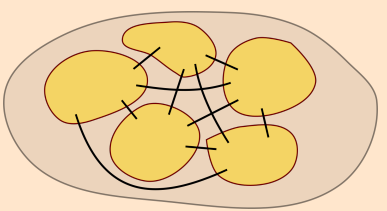

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Key: notion of **embedding**.

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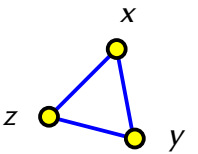

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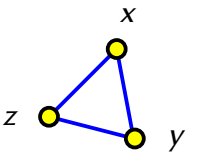

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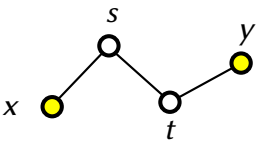

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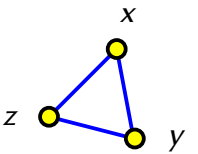

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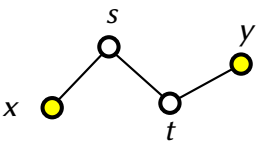

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Every red vertex is at distance $\leqslant 3$ from a blue vertex:

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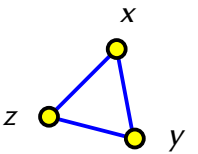

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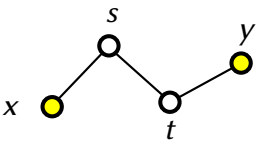

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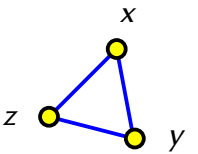

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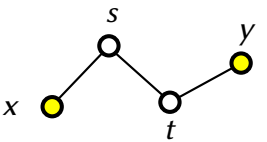

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MSO_2: the graph has a Hamiltonian cycle.

$$\text{Ham} = \exists F.\ [\text{conn}(F) \wedge \forall x.\ \exists^{!2} e.\ [e \in F \wedge \text{inc}(x, e)]].$$

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Model-checking problem for logic $\mathcal{L}$

Given a graph G and a sentence $\varphi \in \mathcal{L}$, is φ true in G?

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Precisely: For what classes $\mathscr{C}$, FO-MC can be solved in fpt time $\mathcal{O}_\varphi(n^c)$?

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Precisely: For what classes $\mathscr{C}$, FO-MC can be solved in fpt time $\mathcal{O}_\varphi(n^c)$?

This is an **excuse** to understand those classes.

Treewidth, rankwidth, and compositionality

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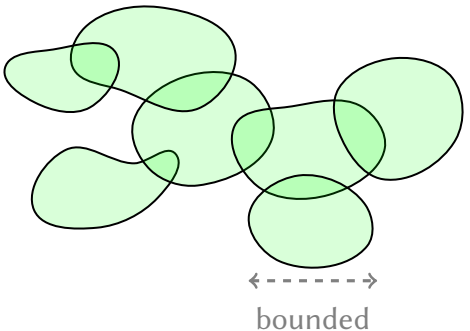
Treewidth:



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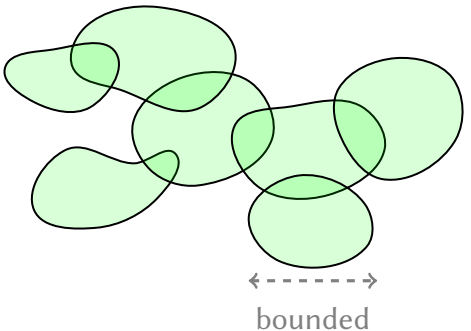

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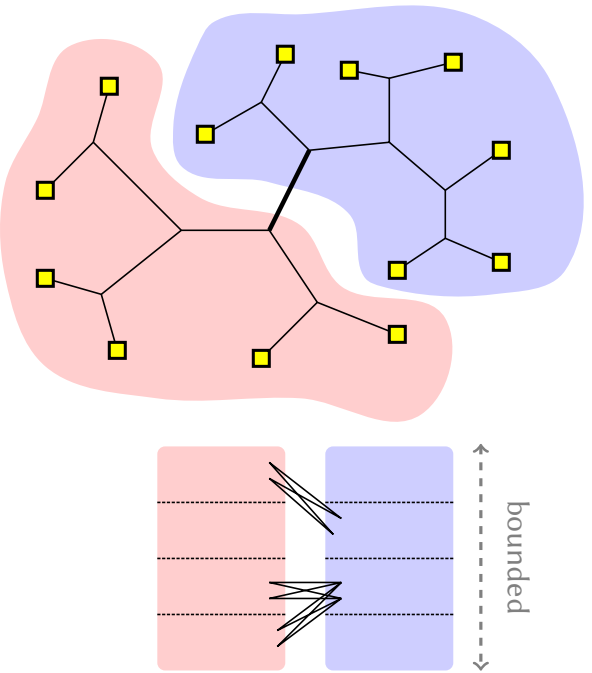

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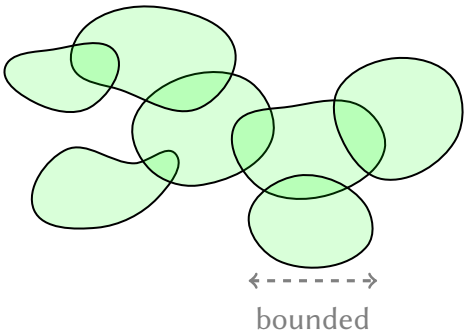

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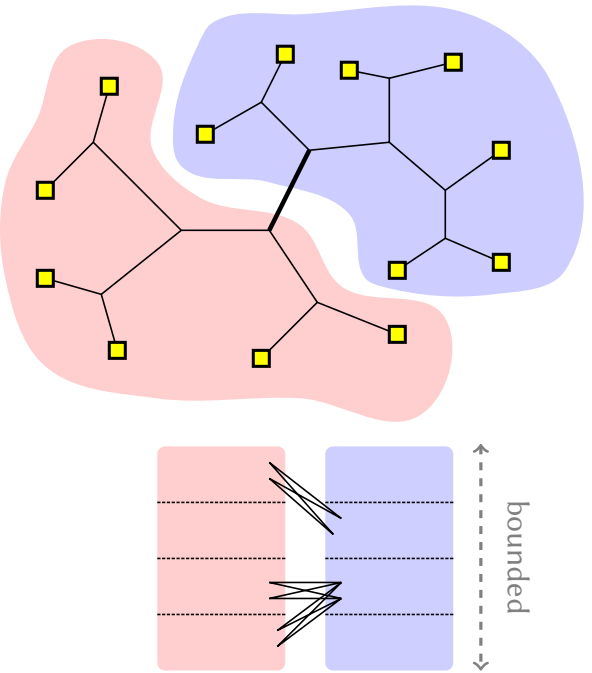

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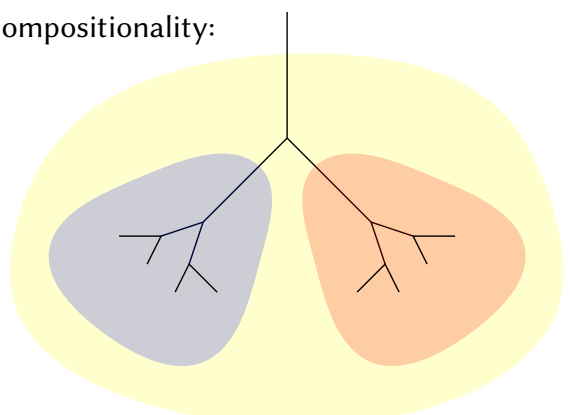

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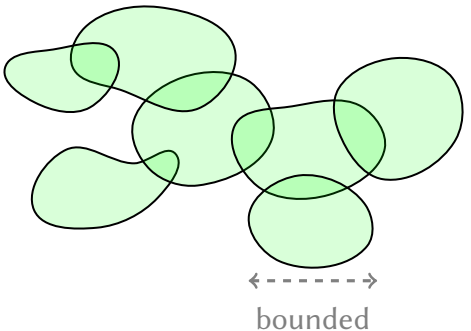

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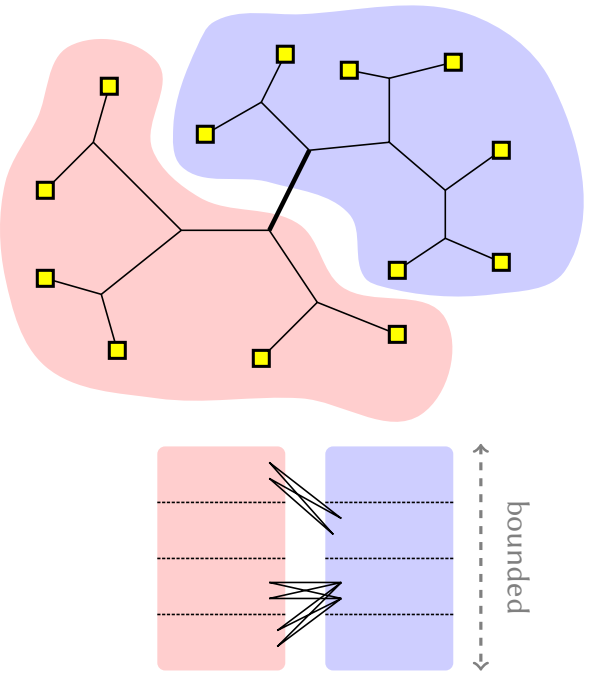

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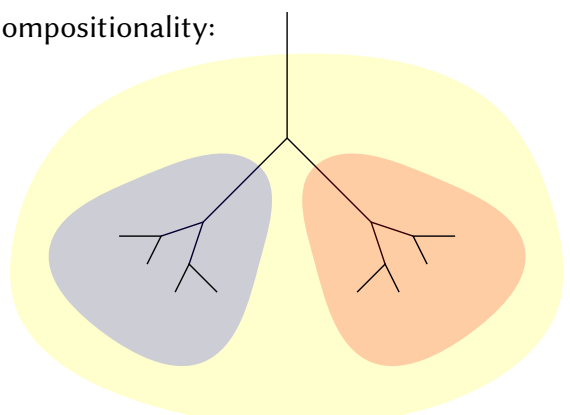

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Model-checking: Compute types bottom-up

Locality

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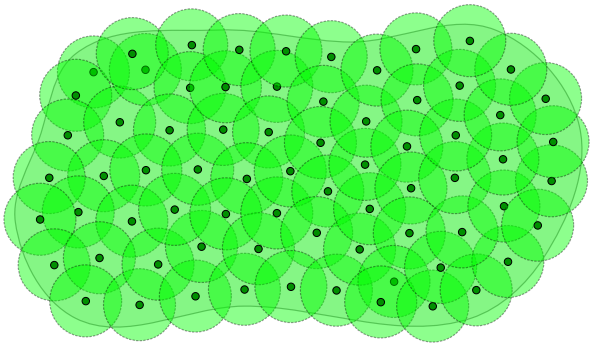

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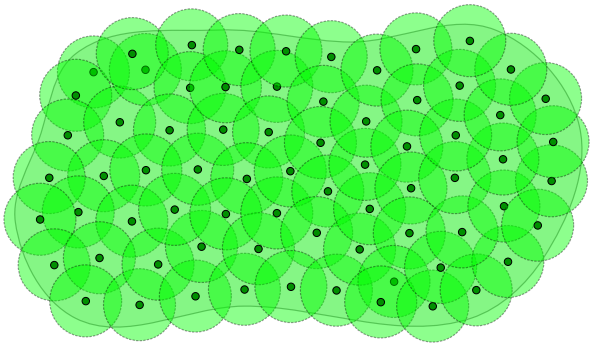

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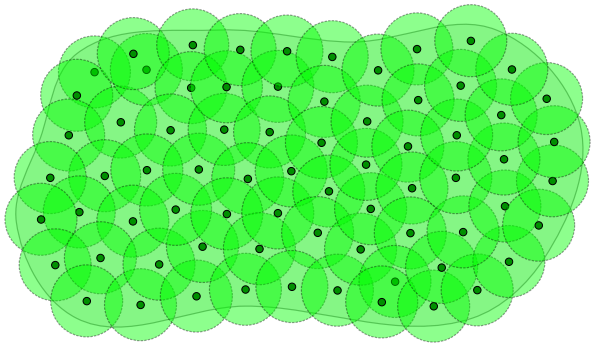

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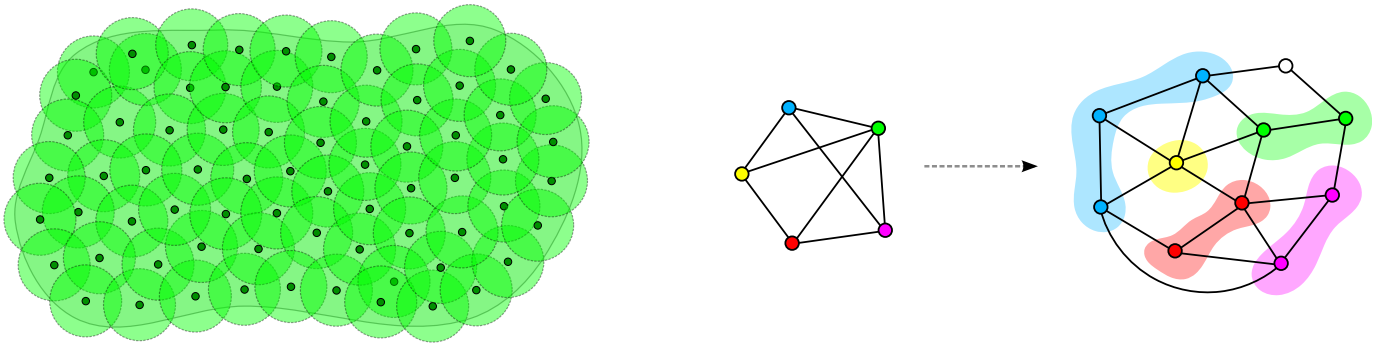

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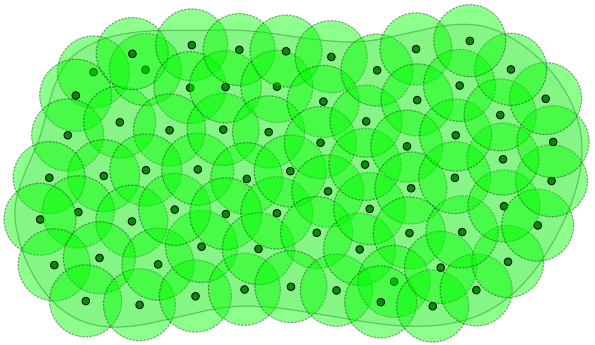

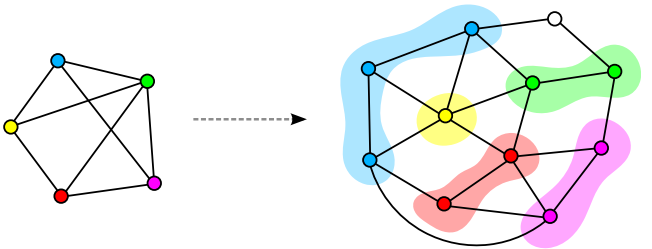

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Thm: H-minor-free graphs admit tree decompositions into almost embeddable parts. [Robertson, Seymour]

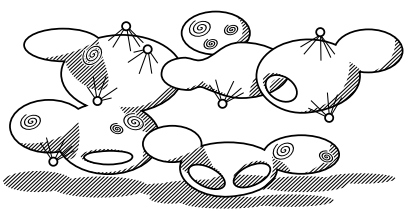

fig. by Felix Reidl

Shallow minors

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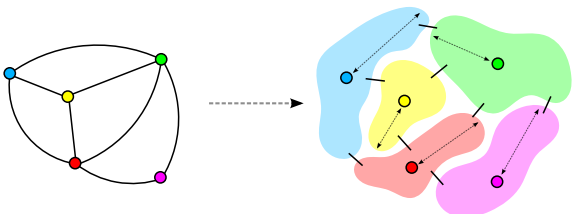

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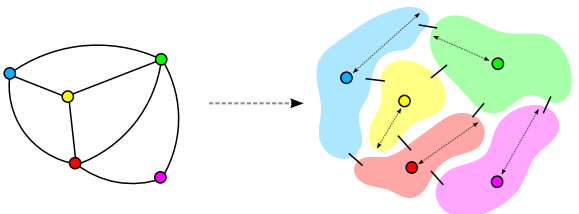

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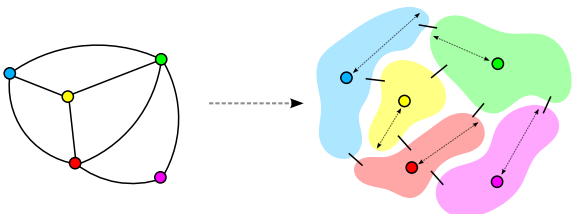

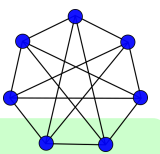

Definition

A graph class $\mathscr{C}$ has **bounded expansion** if for each $d \in \mathbb{N}$, there is $c(d)$ s.t. all depth-d minors of graphs from $\mathscr{C}$ have **avg degree** at most $c(d)$.

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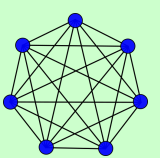

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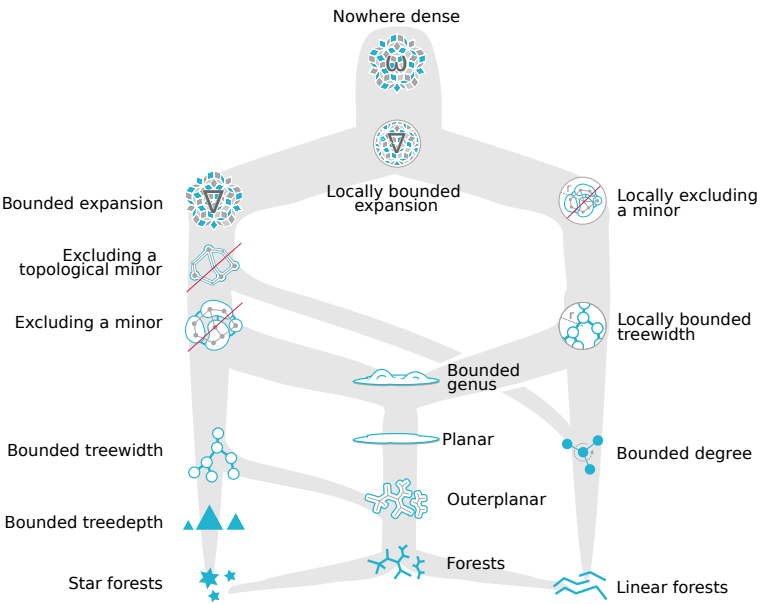

Figure by Felix Reidl

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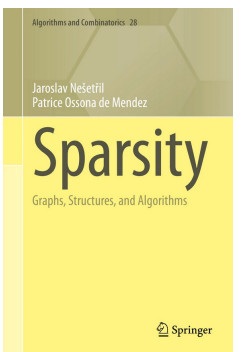

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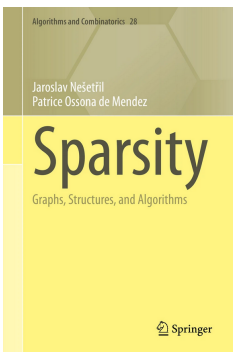

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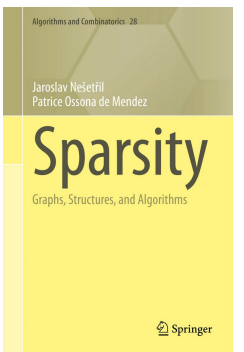

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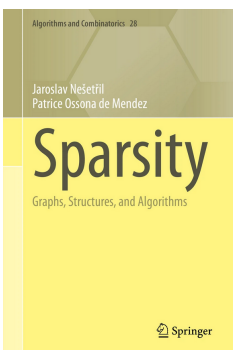

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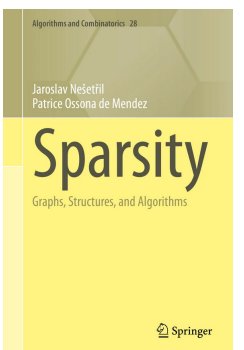

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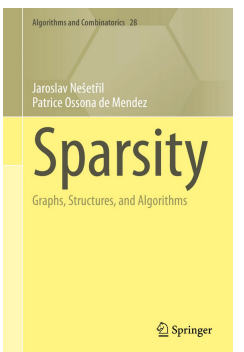

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Sparsity is a **limit property** of a class.

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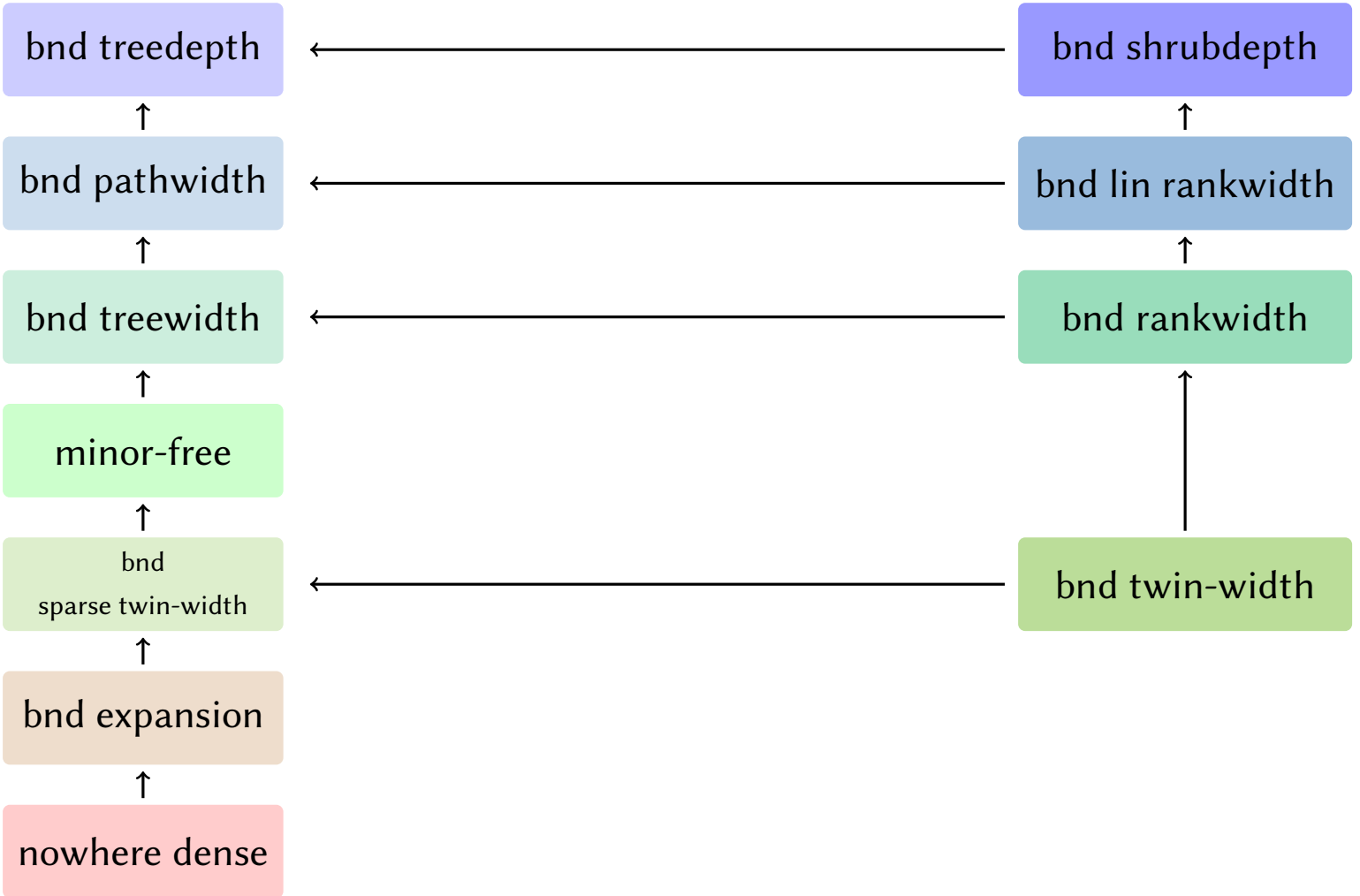
- classes of bounded **rankwidth** or **twin-width**.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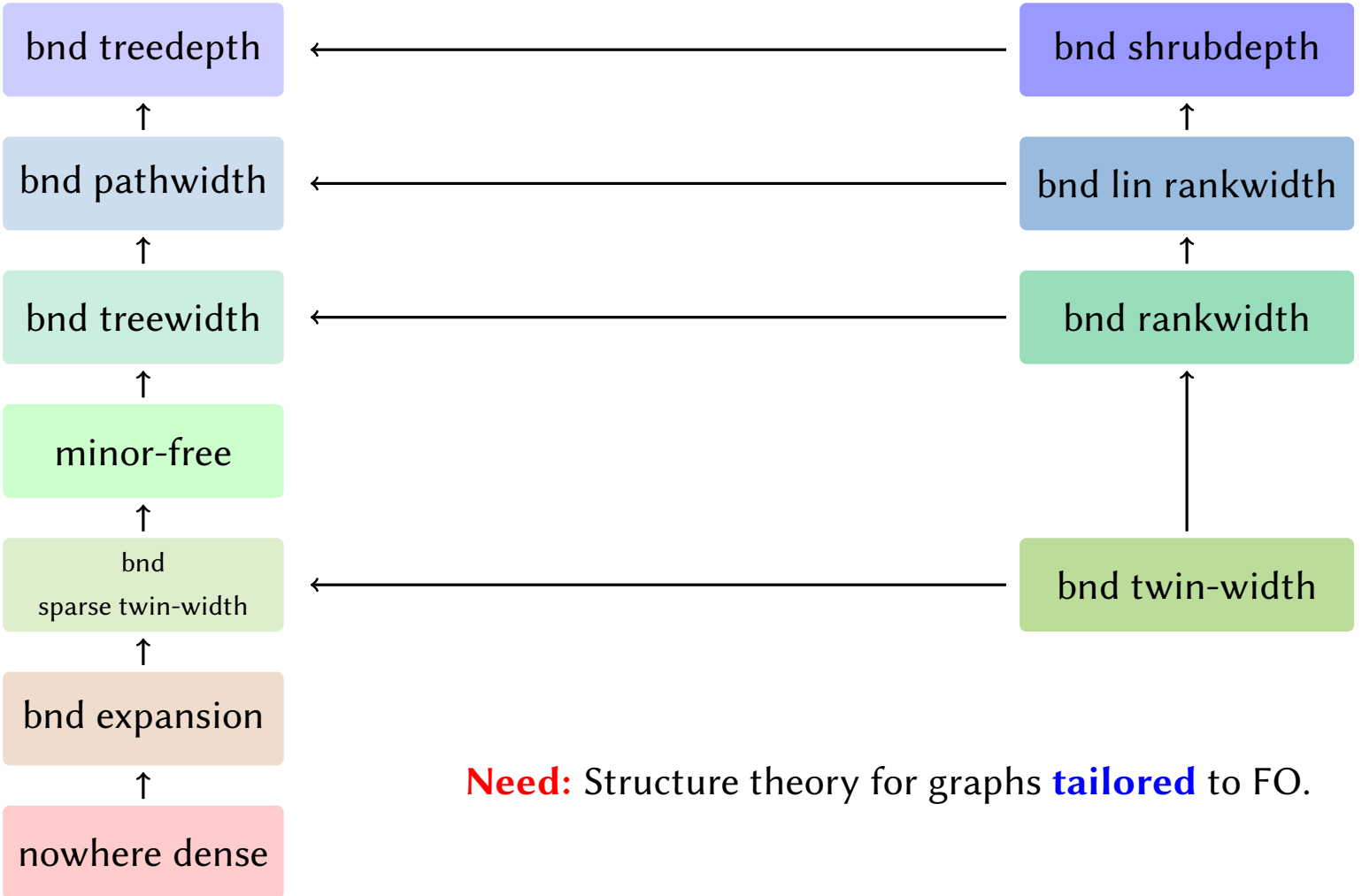
nowhere dense



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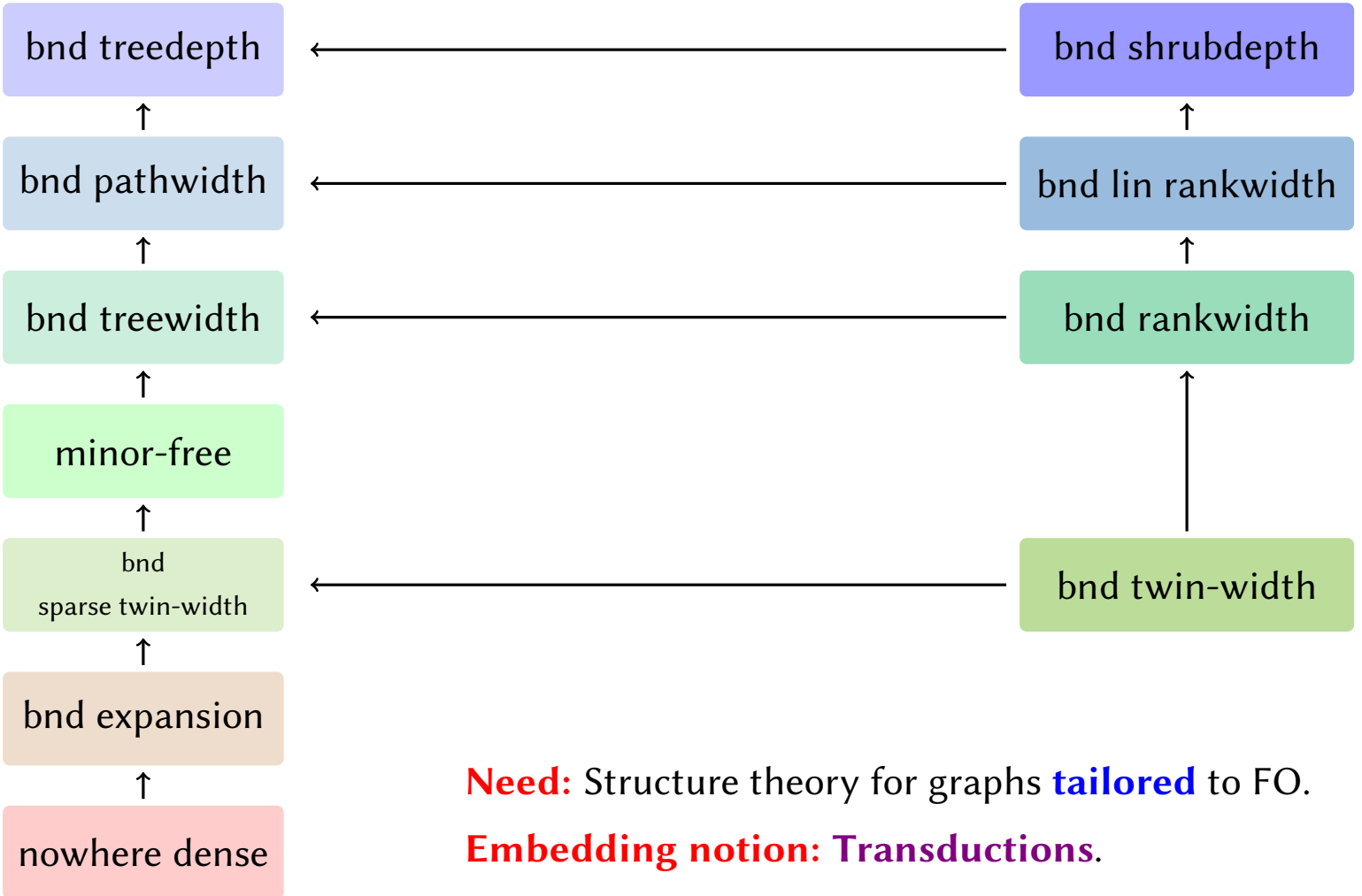

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Embedding notion: Transductions.

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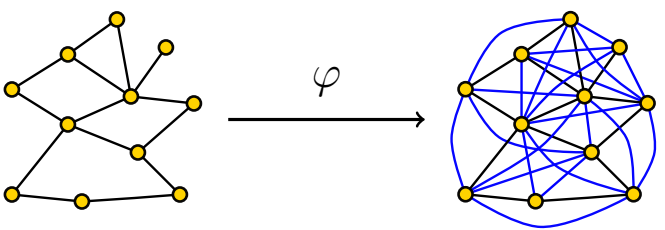

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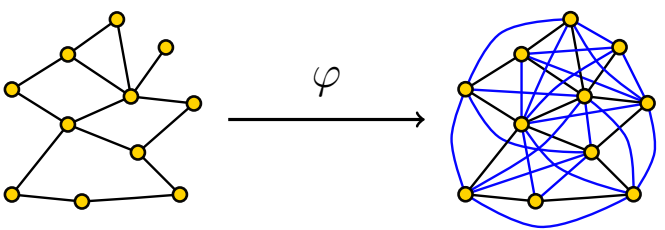

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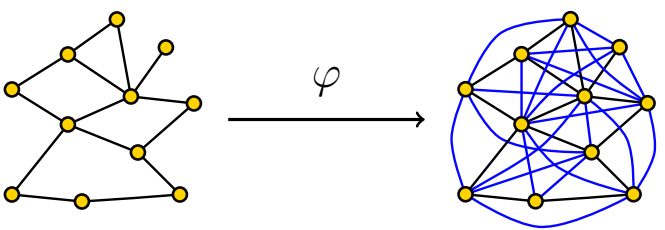

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Obs: Model-checking ψ on $\varphi(G)$ **reduces** to

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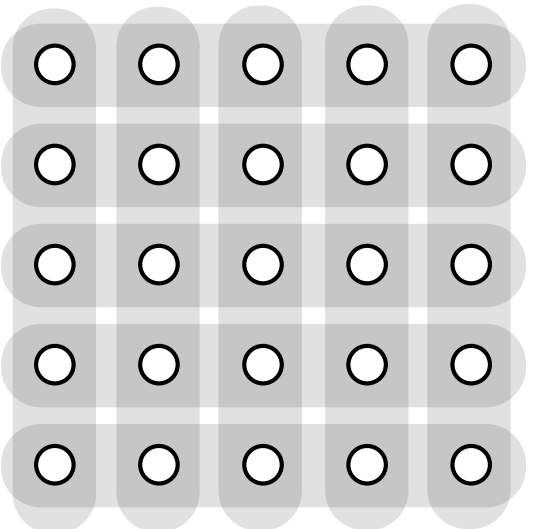

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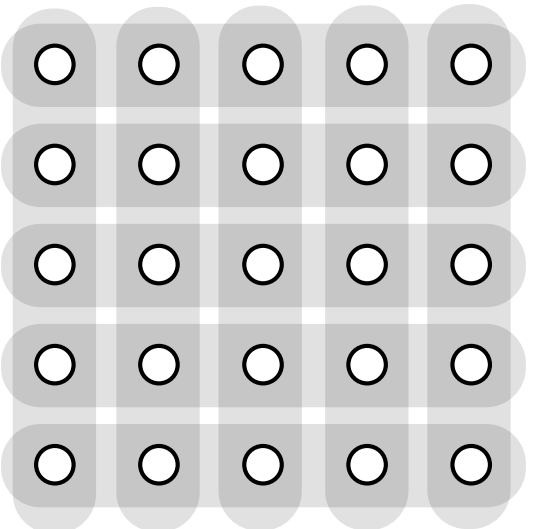

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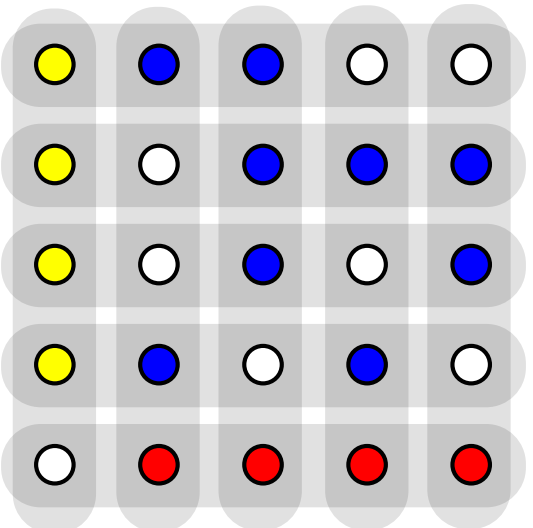

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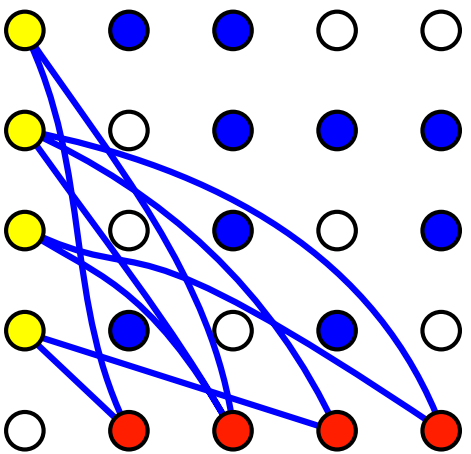

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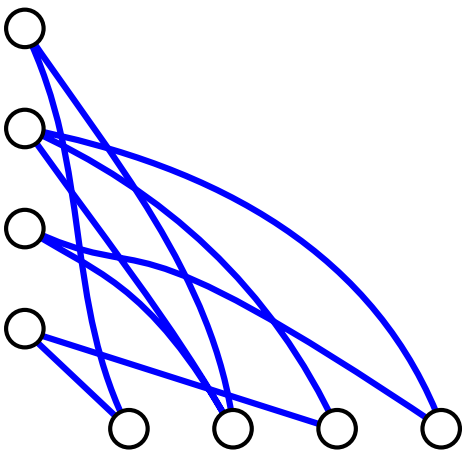

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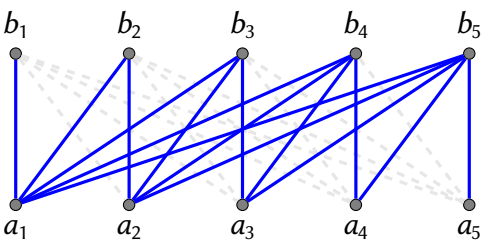

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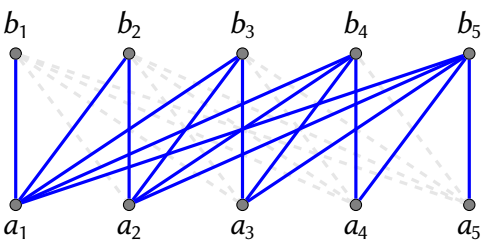

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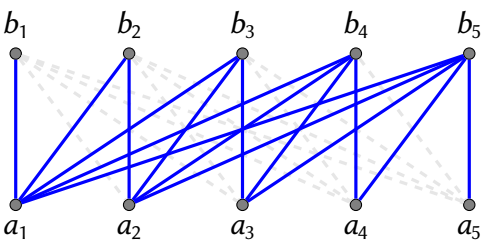

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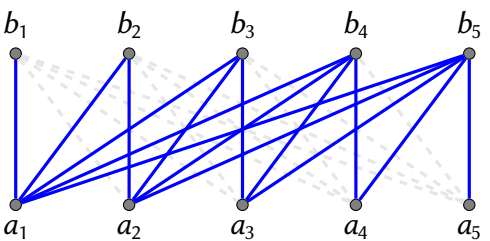

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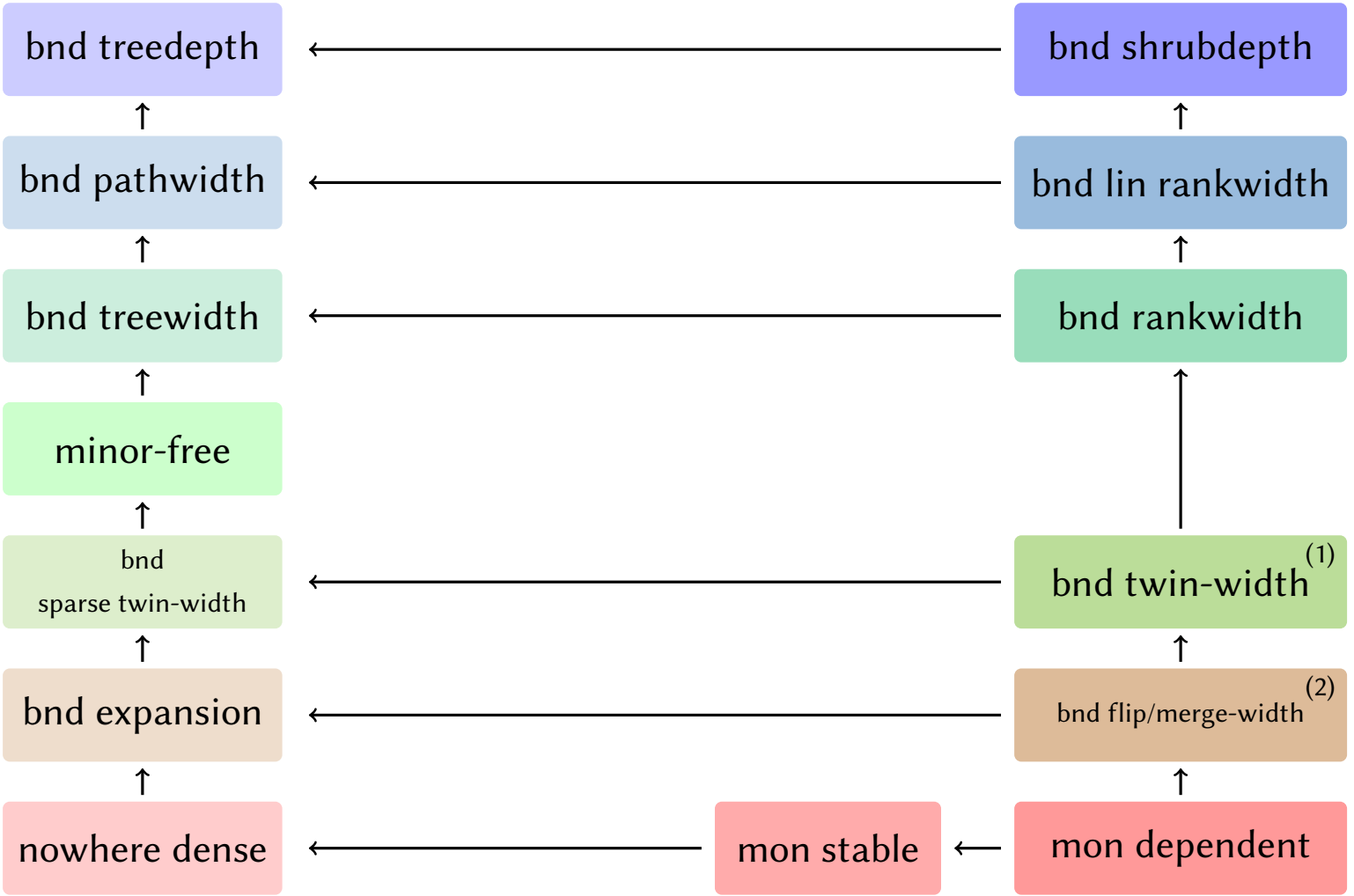
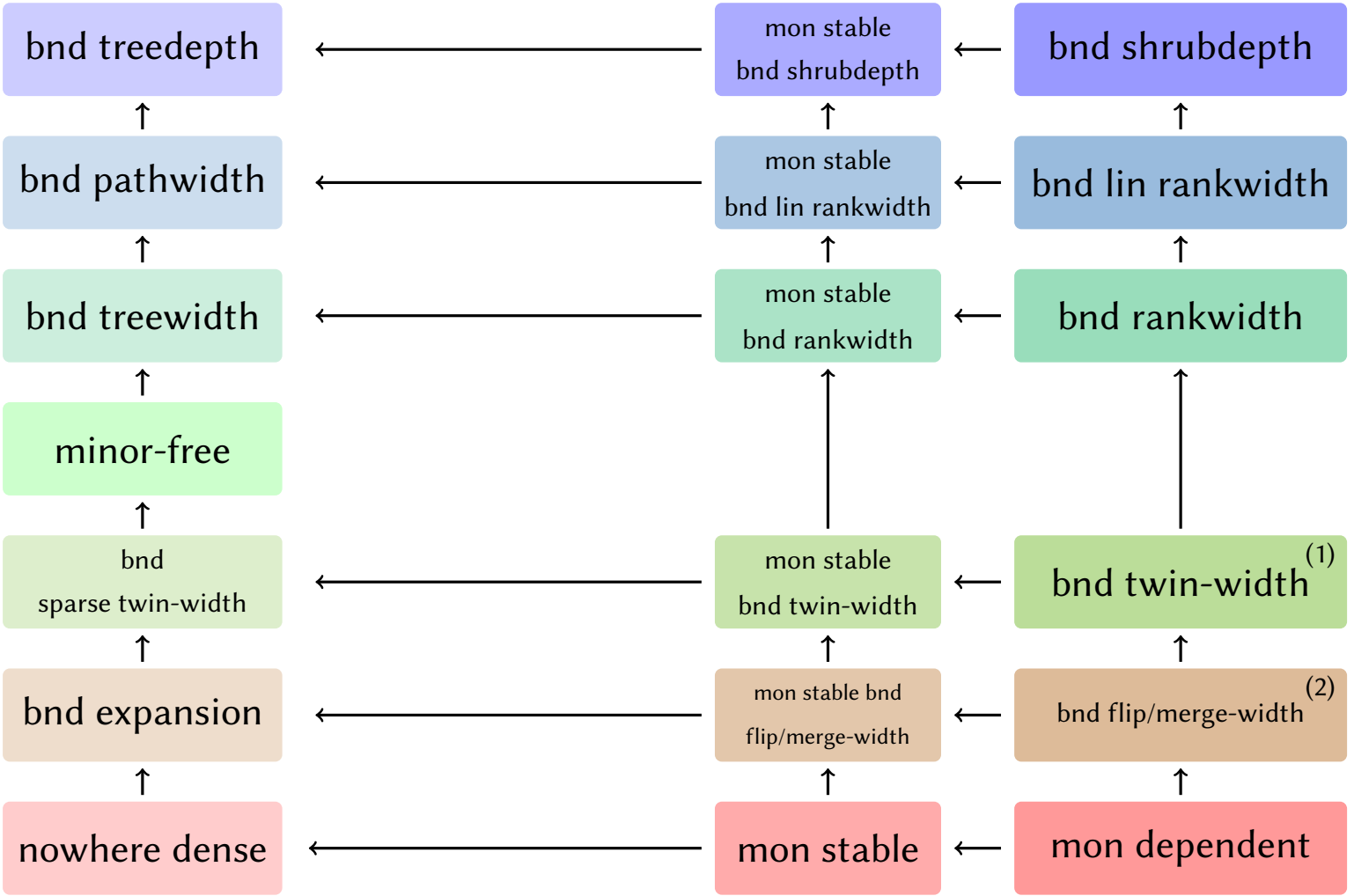


(1): Bonnet, Kim, Thomassé, Watrigant; '20
(2): Toruńczyk; '23 Dreier, Toruńczyk; '25^{+}



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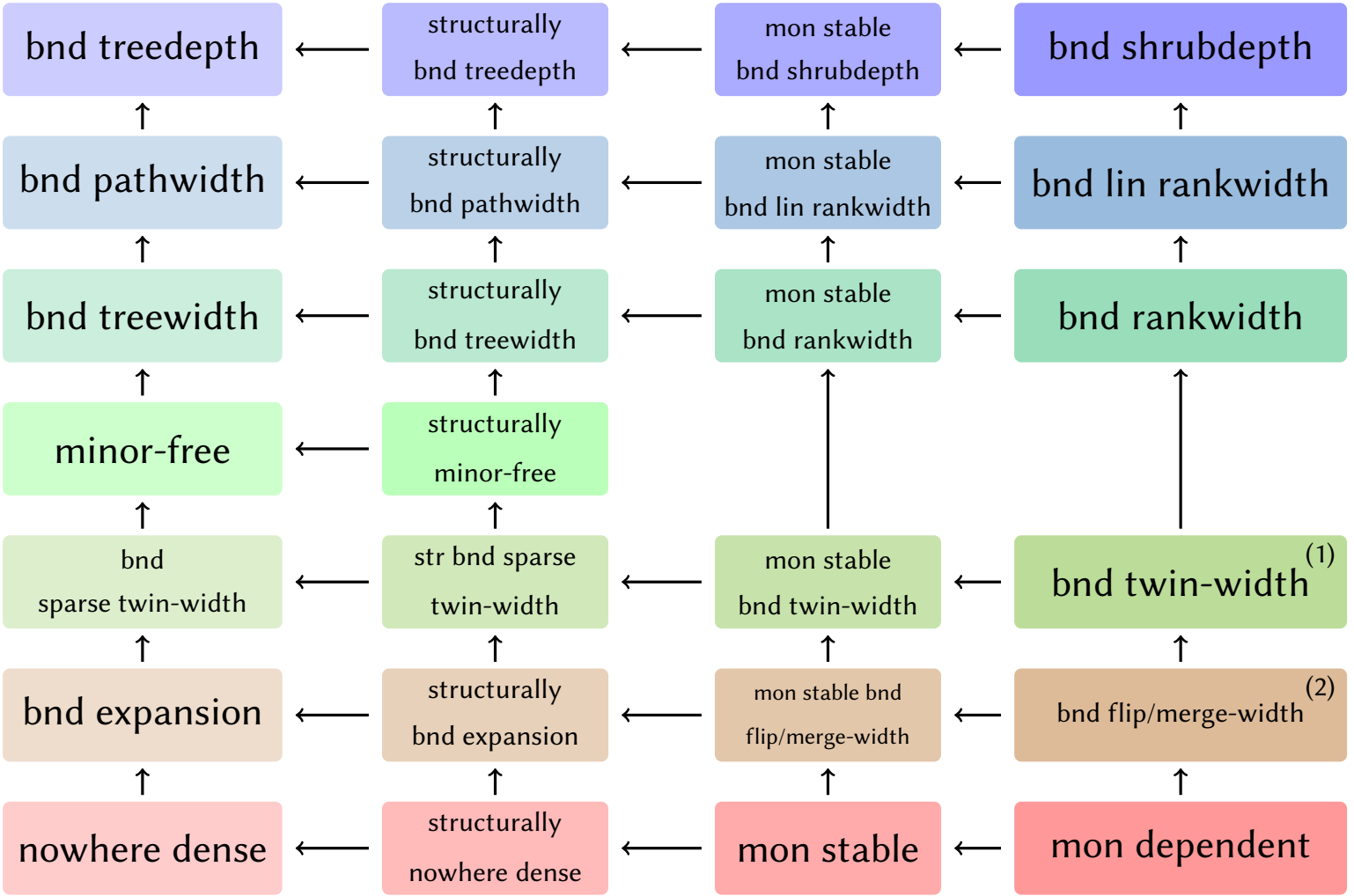

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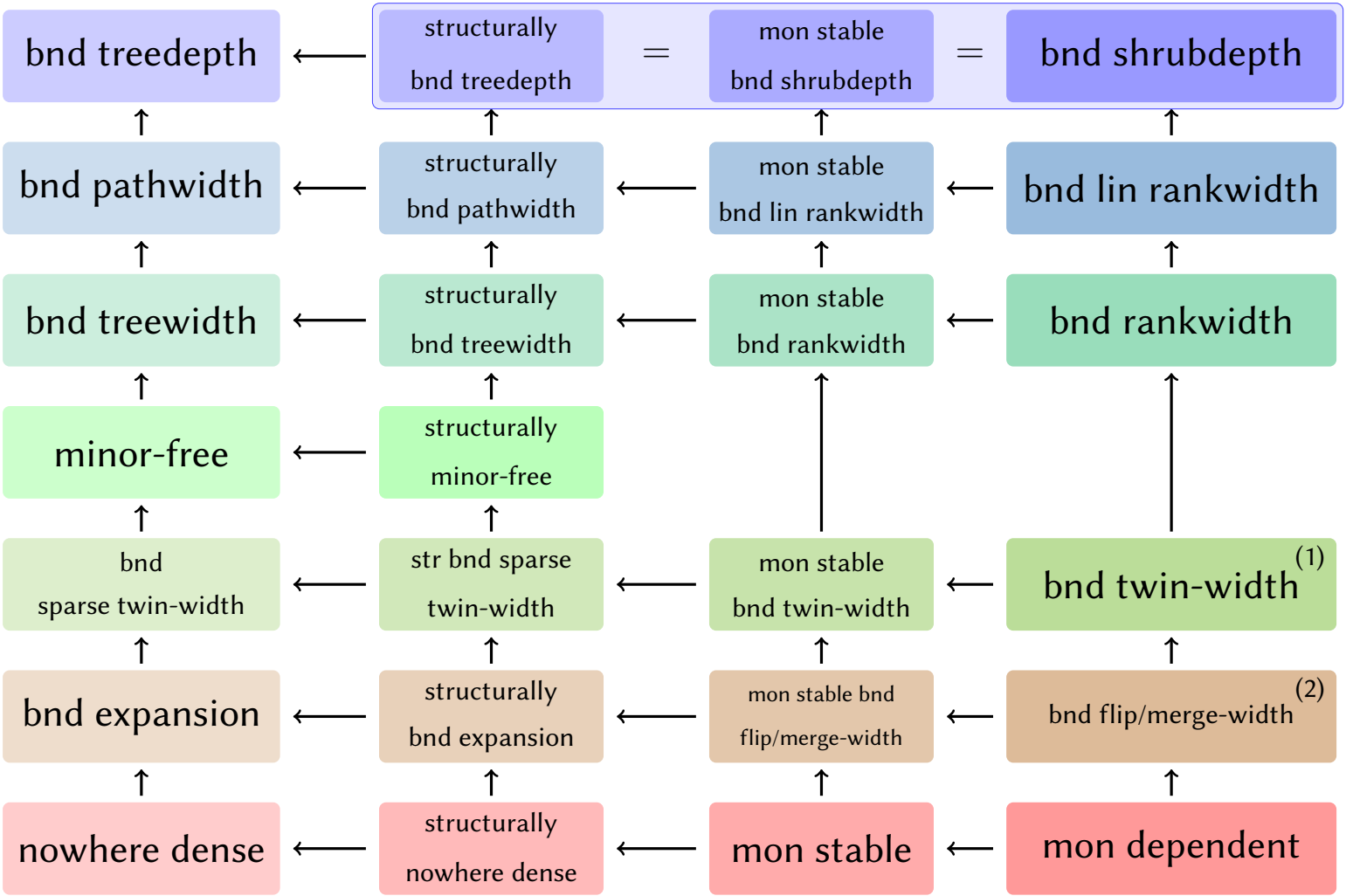

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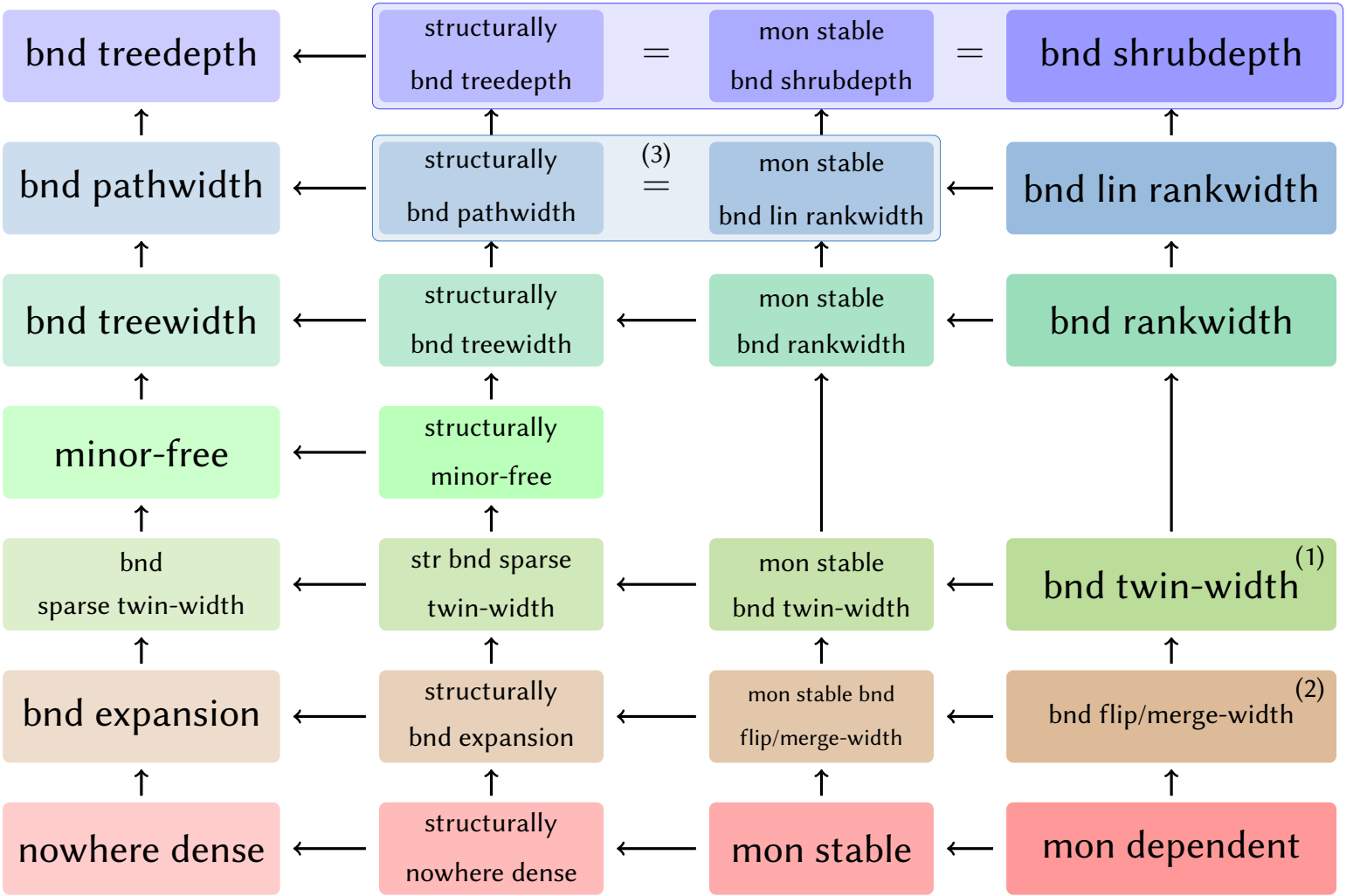

(1): Bonnet, Kim, Thomassé, Watrigant; '20 (3): Nešetřil, Ossona de Mendez, Rabinovich, Siebertz; '20
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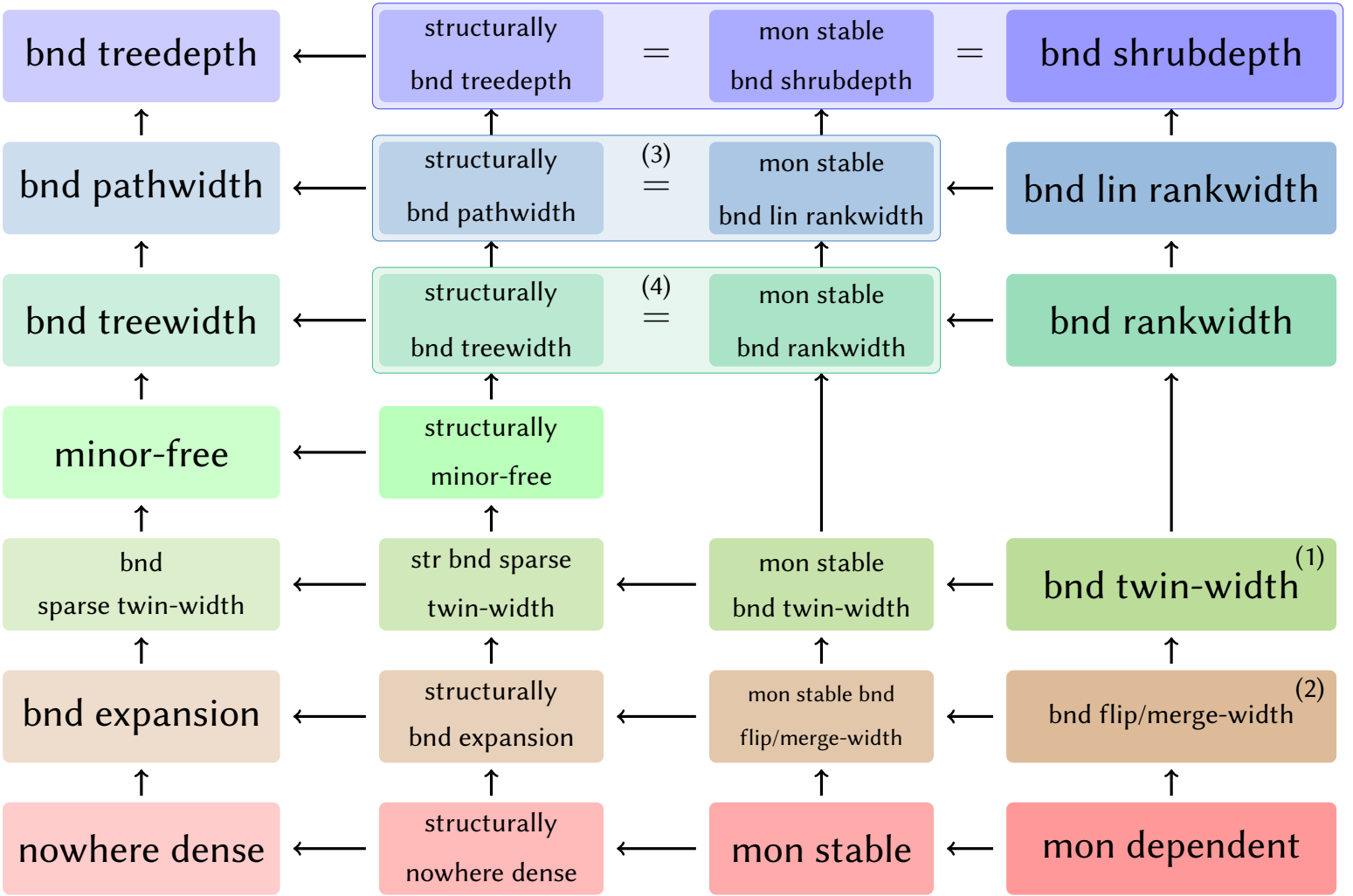

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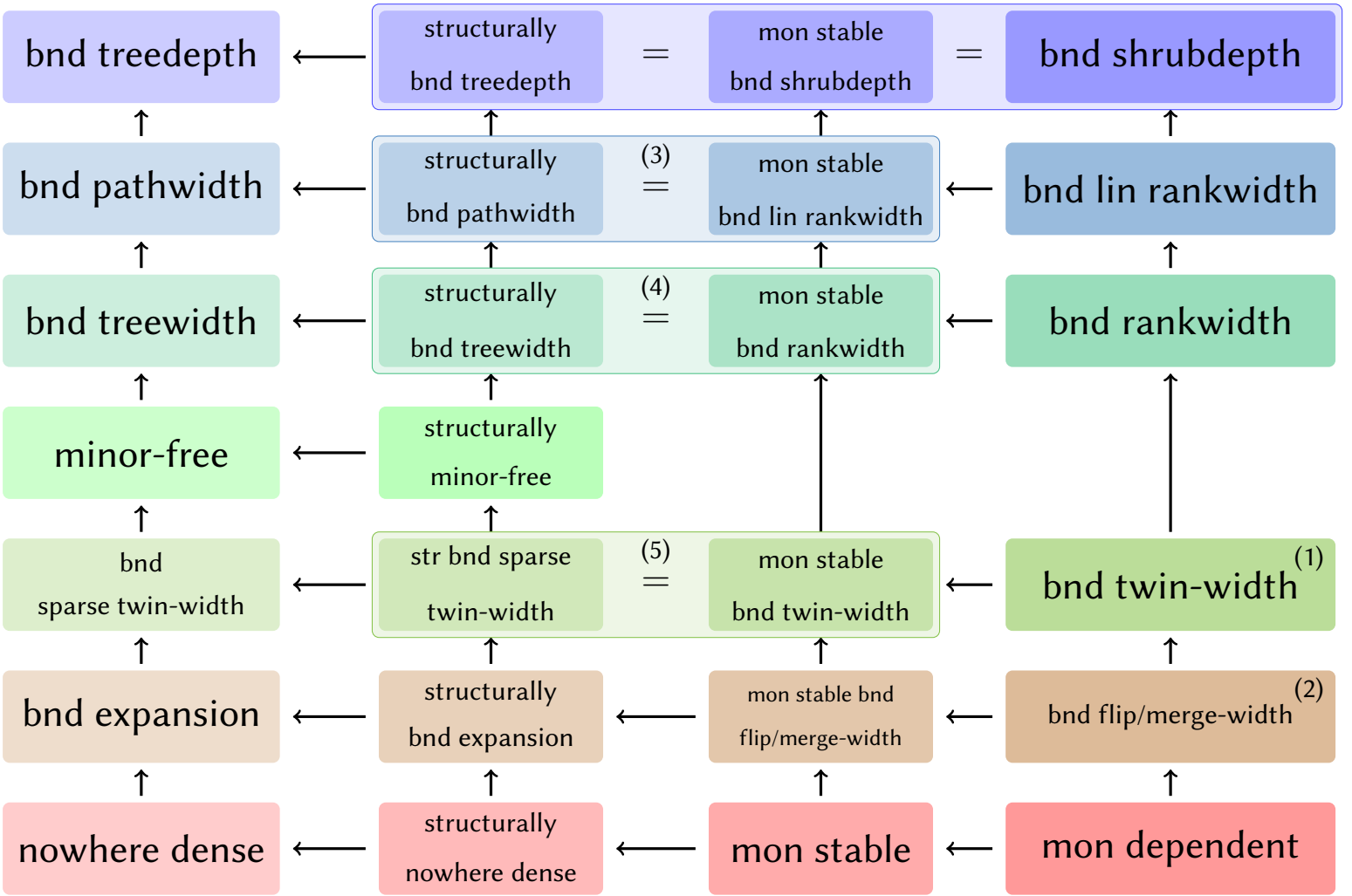
(4): Nešetřil, Ossona de Mendez, P, Rabinovich, Siebertz; '21



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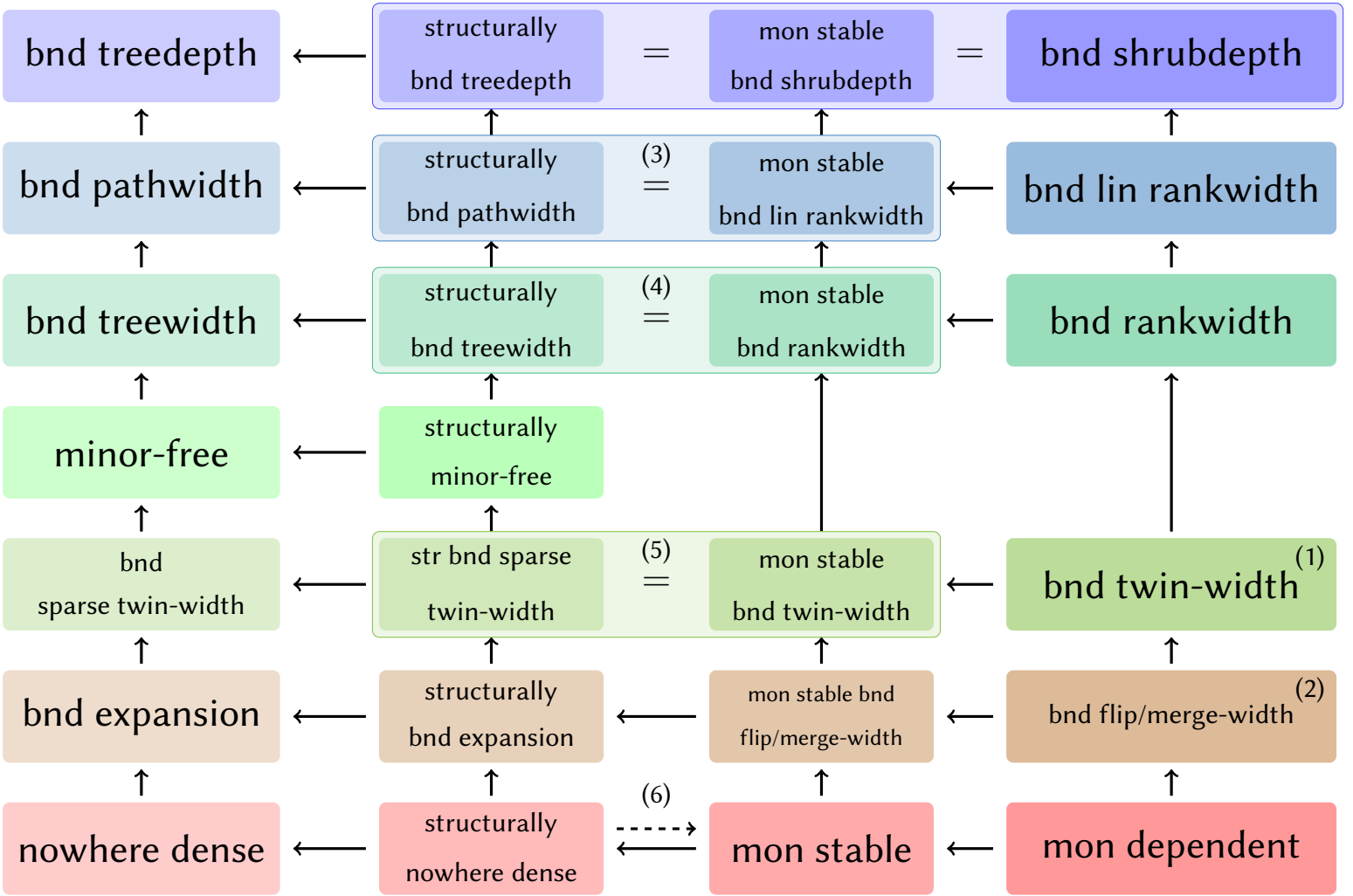
(5): Gajarský, P, Toruńczyk; '22



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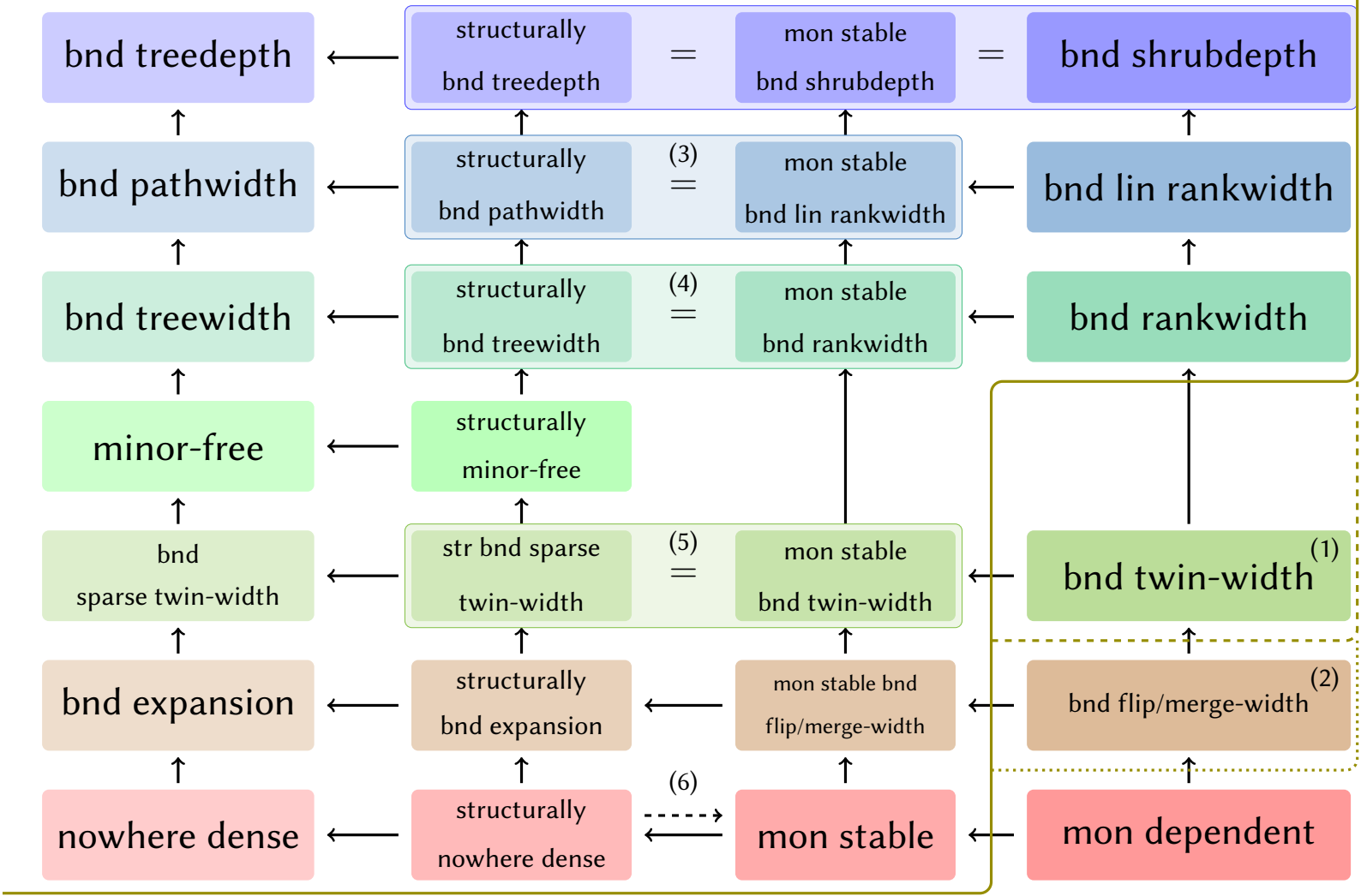

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Tools

mon stable

mon dependent

Tools

mon stable	mon dependent
Separators	
flip-flatness (1)	flip-breakability (2)

(1): Dreier, Mählmann, Toruńczyk; ’23
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Tools

mon stable	mon dependent
Separators	
flip-flatness (1)	flip-breakability (2)
Decompositions	
Flipper Game (3)	?

(1): Dreier, Mählmann, Toruńczyk; ’23
(2): Dreier, Mählmann, Toruńczyk; ’24
(3): GMMcCOPPSST; ’23

Tools

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	Separators	
flip-flatness (1)		flip-breakability (2)
	Decompositions	
Flipper Game (3)		?
	Structure of neighborhoods	
VC density $1+\varepsilon$ (4)		
$\Rightarrow$ Welzl orders		
$\Rightarrow$ Neighborhood covers		

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Tools

mon stable		mon dependent
Separators		
flip-flatness [1]		flip-breakability [2]
Decompositions		
Flipper Game [3]		?
Structure of neighborhoods		
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$\Rightarrow$ Welzl orders		$\Rightarrow$ Welzl orders
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Tools

mon stable		mon dependent
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Obstructions		
Patterns under flips [2,4]		Patterns under flips [2]
Shallow vertex-minors [6]		Shallow vertex-minors [6]

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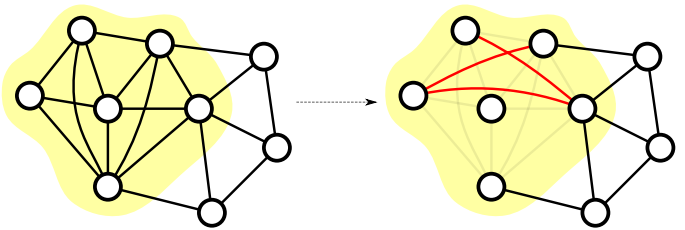
Flip-flatness and flip-breakability

Flip-flatness and flip-breakability

Flip/Perturbation: Complement the edge relation on a vertex subset.



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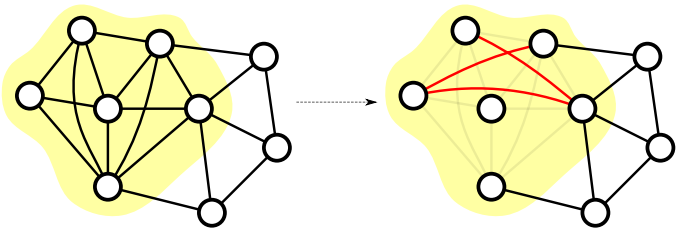
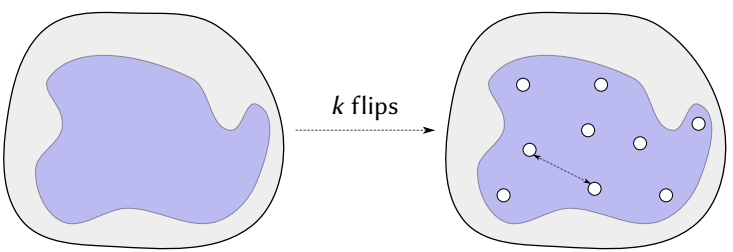


Def: $\mathscr{C}$ is **flip-flat** if $\forall d\, \exists k$ so that $\forall G \in \mathscr{C}\, \forall W \subseteq V(G)$, there is $A \subseteq W$ of size $\mathcal{U}(|W|)$ that can be made d-scattered by applying k flips.



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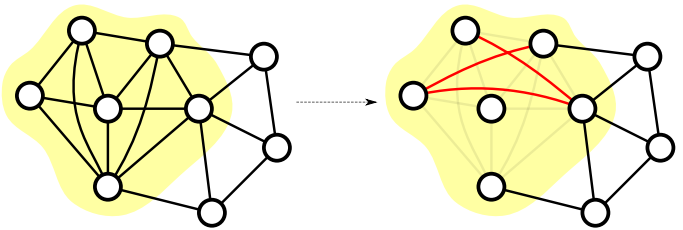

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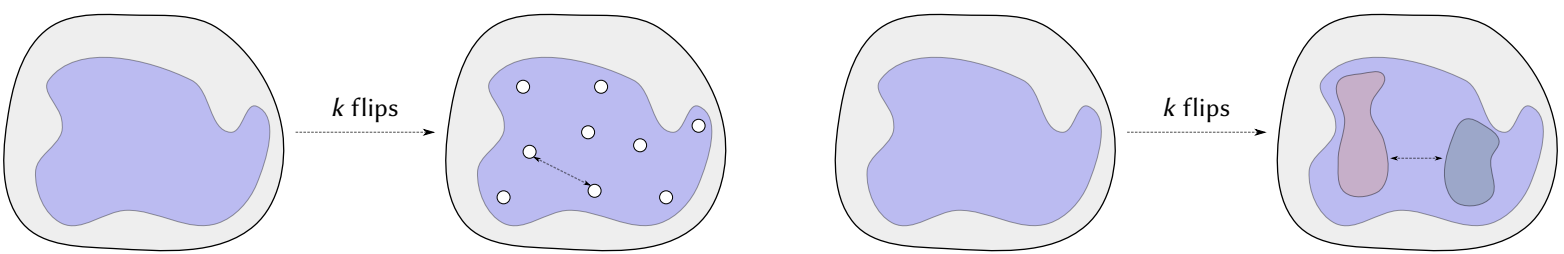
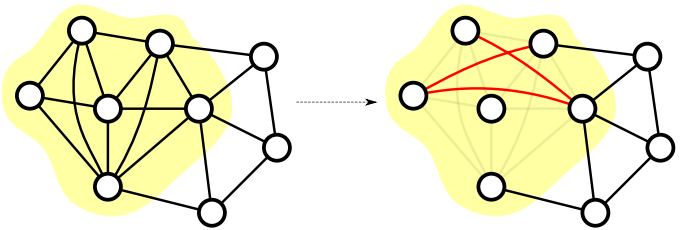
Def: $\mathscr{C}$ is **flip-breakable** if $\forall d\, \exists k$ so that $\forall G \in \mathscr{C}\, \forall W \subseteq V(G)$, there are $A, B \subseteq W$ of size $\mathcal{U}(|W|)$, so that $\mathrm{dist}(A, B) > d$ after applying k flips.



Flip-flatness and flip-breakability

Flip/Perturbation: Complement the edge relation on a vertex subset.



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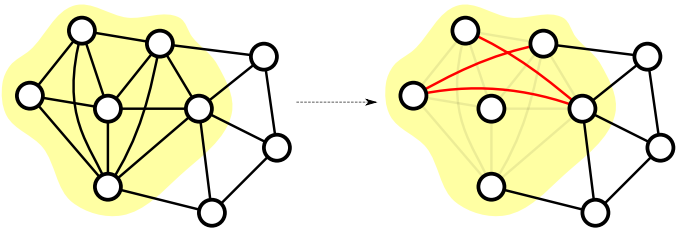
Theorem [Dreier, Mählmann, Toruńczyk; '23, '24]

Monadic stability $\Leftrightarrow$ Flip-flatness

Monadic dependence $\Leftrightarrow$ Flip-breakability

Flip-flatness and flip-breakability

Flip/Perturbation: Complement the edge relation on a vertex subset.



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Theorem [Dreier, Mählmann, Toruńczyk; '23, '24]

Monadic stability $\Leftrightarrow$ Flip-flatness

Monadic dependence $\Leftrightarrow$ Flip-breakability

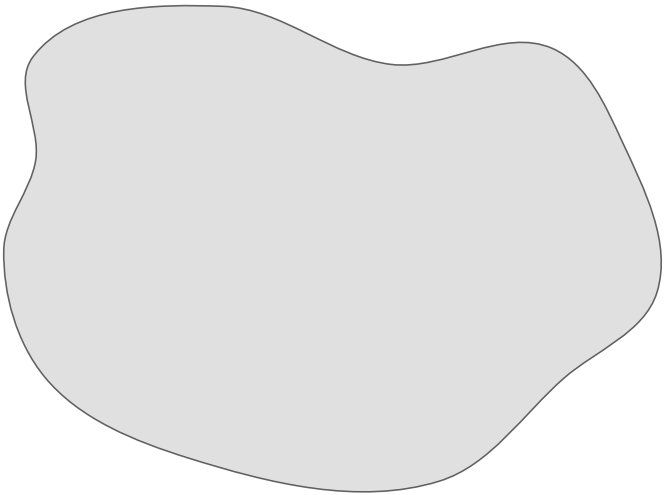
Fact: Flatness $\Leftrightarrow$ Breakability $\Leftrightarrow$ Nowhere denseness,

where **flips** are replaced with **vertex deletions**.

Flipper game

Flipper game

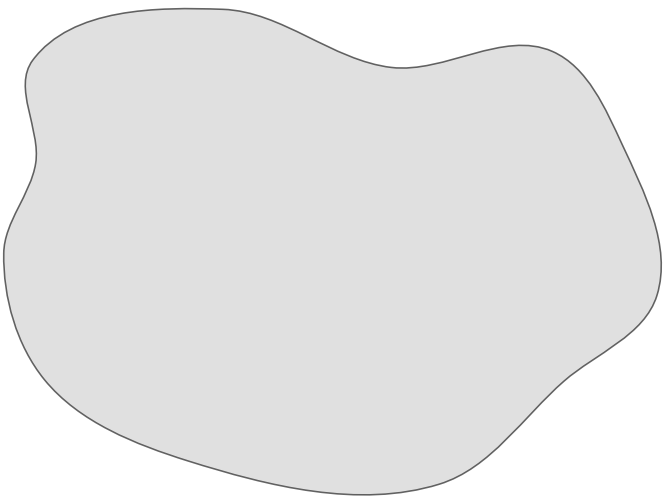
Fixed radius $d \in \mathbb{N}$. Played on a graph G.



Flipper game

Fixed radius $d \in \mathbb{N}$. Played on a graph G.

Two players: **Flipper** and **Connector**

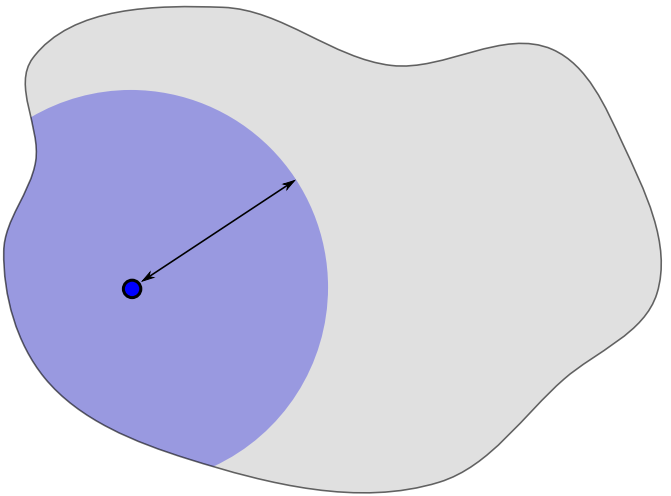


Round

Flipper game

Fixed radius $d \in \mathbb{N}$. Played on a graph G.

Two players: **Flipper** and **Connector**



Round

Connector picks a ball of radius d.

Flipper game

Fixed radius $d \in \mathbb{N}$. Played on a graph G.

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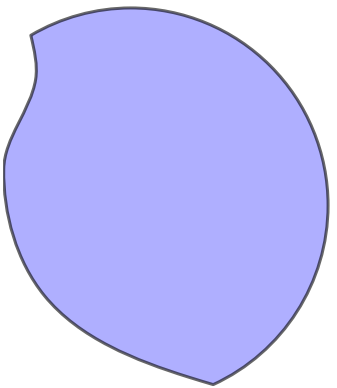


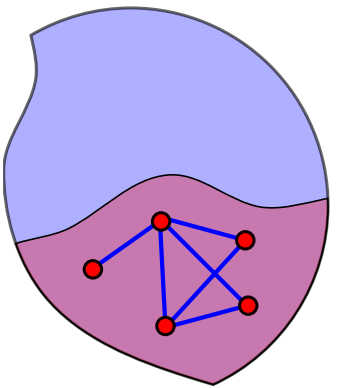
Round

Connector picks a ball of radius d. $\rightsquigarrow$ The arena is **restricted** to this ball.

Flipper game

Fixed radius $d \in \mathbb{N}$. Played on a graph G.

Two players: **Flipper** and **Connector**



Round

Connector picks a ball of radius d. $\rightsquigarrow$ The arena is **restricted** to this ball.

Flipper chooses any **vertex subset**.

Flipper game

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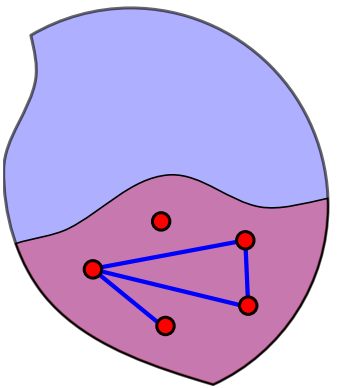

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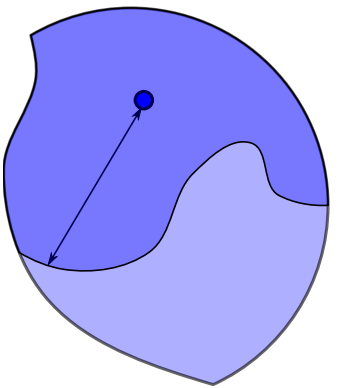

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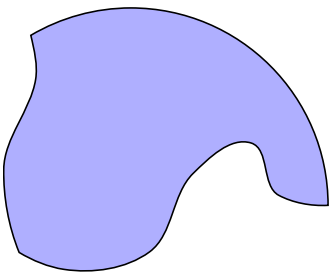

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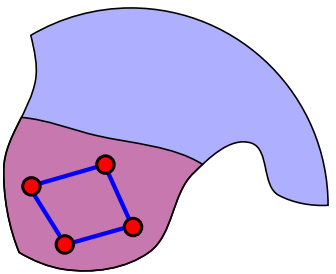

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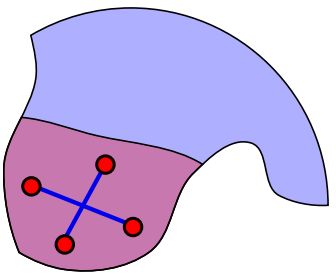

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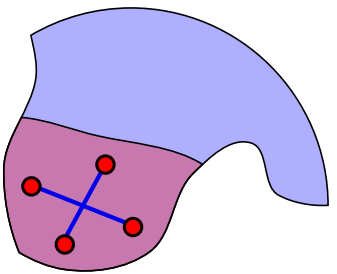

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Game **finishes** when the arena gets reduced to **one vertex**.

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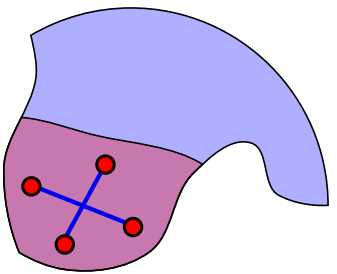

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Game **finishes** when the arena gets reduced to **one vertex**.

Flipper: finish the game quickly **Connector**: stay alive

Flipper game and monadic stability

Flipper game and monadic stability

Splitter Game: Same, but Splitter deletes a vertex in every round.

Flipper game and monadic stability

Splitter Game: Same, but Splitter deletes a vertex in every round.

Theorem [Grohe, Kreutzer, Siebertz; '14]

$\mathscr{C}$ is **nowhere dense**. $\Leftrightarrow$

For every $d \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that **Splitter** can win the radius-d Splitter game in any $G \in \mathscr{C}$ in $\leqslant k$ rounds.

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Game tree $=$ **Shallow decomposition**

Flipper game and monadic stability

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Game tree $=$ **Shallow decomposition**

vertex deletion $\rightsquigarrow$ **flip**

Structure of neighborhoods

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Theorem [Dreier, Eleftheriadis, Mählmann, McCarty, P, Toruńczyk; '24]

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$\mathscr{C}$ **mon stable**. Then for every $G \in \mathscr{C}$ and $d \in \mathbb{N}$, there is $\mathcal{F} \subseteq 2^{V(G)}$ s.t.:

- every $A \in \mathcal{F}$ has weak diameter $\leqslant 4d$;
- every radius-d ball B is contained in some $A \in \mathcal{F}$;
- every vertex v belongs to $\mathcal{O}_\varepsilon(n^\varepsilon)$ sets $A \in \mathcal{F}$.

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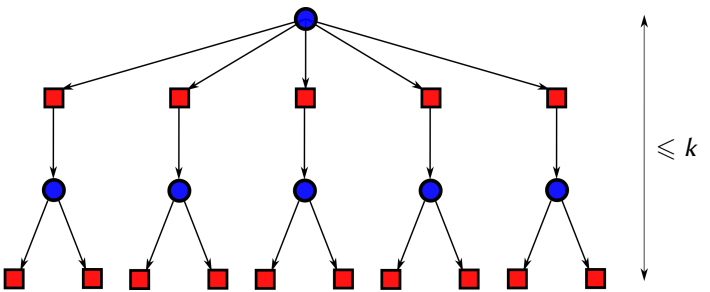

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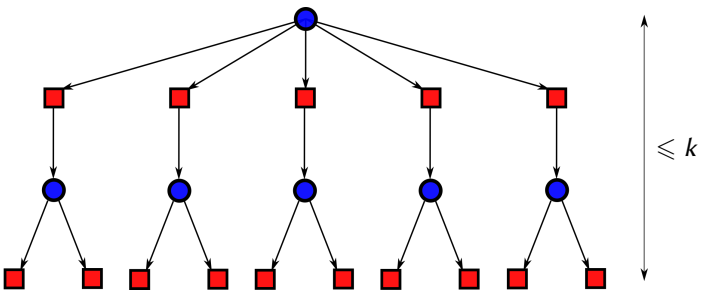

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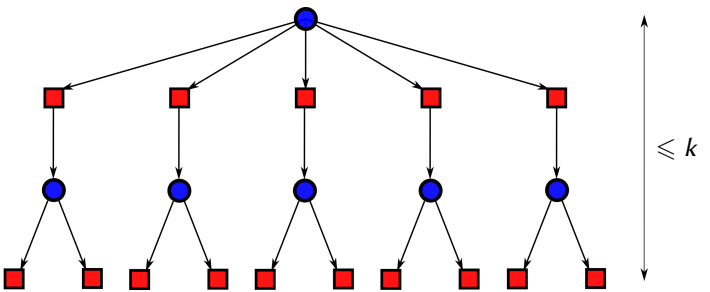

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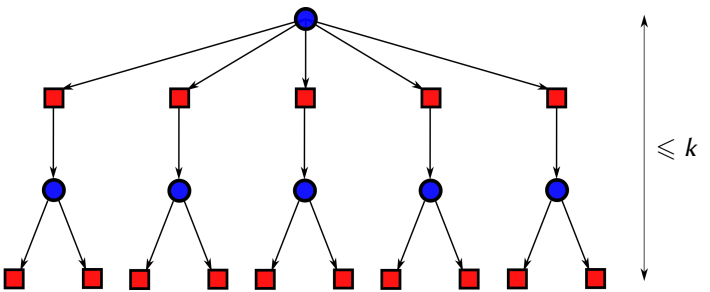

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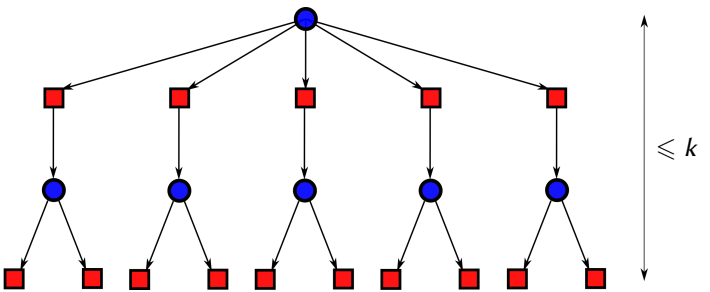

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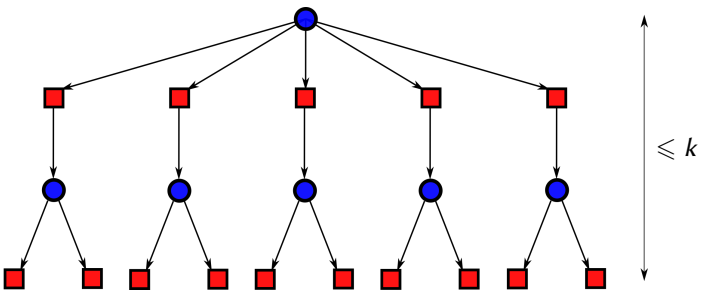

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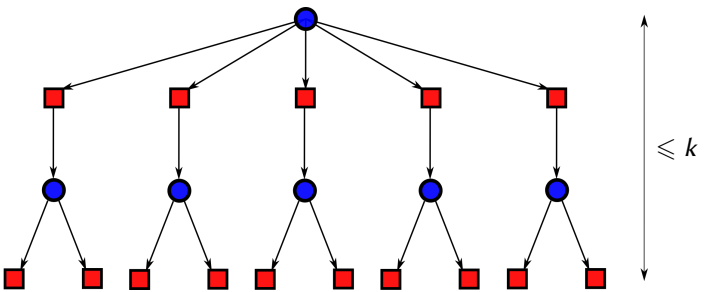

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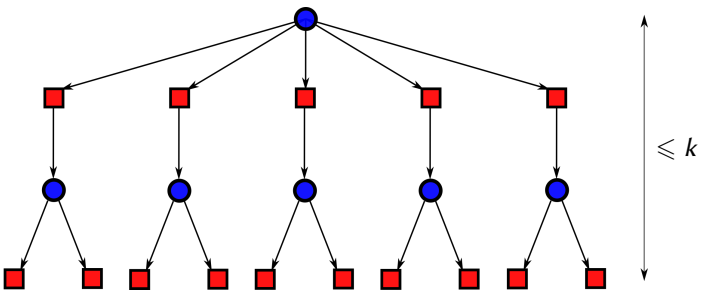

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$\mathscr{C}$ is **mon dependent** iff there are no $r, k \in \mathbb{N}$ s.t. $\mathscr{C}$ contains induced:

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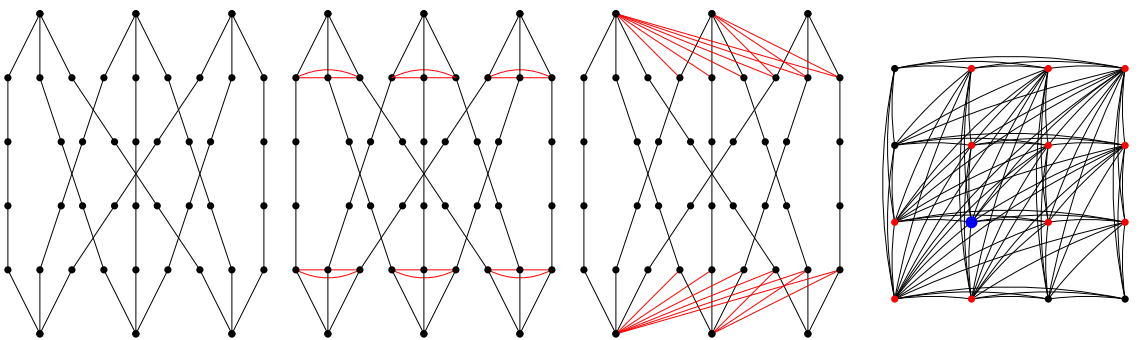

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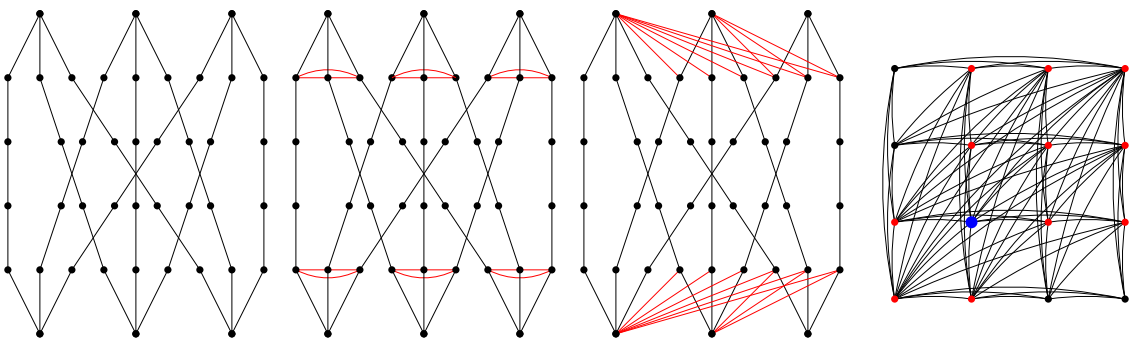

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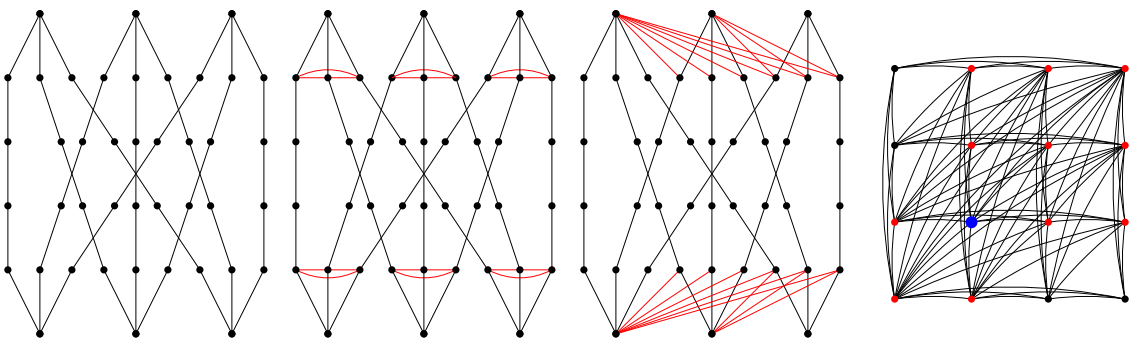

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Fact: For **mon stable**: star crossings, clique crossings, and half-graphs.

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Thanks for attention!