Characterization of stability by quasibounded-size bounded-shrubdepth decompositions

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# $\Pi$ -Decompositions



### $\Pi$ -Decompositions

 $\Pi: \text{ hereditary class property, } f: \mathbb{N} \to \mathbb{N}, \ p \in \mathbb{N}.$ 

#### Definition

 $\mathscr{C}$  has an *f*-bounded  $\Pi$ -decomposition with parameter *p* if there exists  $\mathscr{D}_p \in \Pi$  such that

$$\forall G \in \mathscr{C} \quad \exists \text{ partition } \mathscr{P} \text{ of } V(G) \text{ with } |\mathscr{P}| \leq f(|G|) \text{ and}$$

$$G[P_1\cup\cdots\cup P_p]\in\mathscr{D}_p$$
  $\forall P_1,\ldots,P_p\in\mathscr{P}.$ 

- $f(n) = O(1) \longrightarrow$  bounded-size  $\Pi$ -decomposition;
- $f(n) = n^{o(1)} \longrightarrow$  quasibounded-size  $\Pi$ -decomposition.

### What property $\Pi$ ?



# $\Pi$ -Decompositions

	$f(n)$ $\Pi$	bounded tree depth	bounded shrubdepth
bounded size	<i>O</i> (1)	bounded expansion [NO '08]	structurally bounded expansion [GKNOPST '20].
quasi-bounded size	$n^{o(1)}$	nowhere dense [NO '11]	

### $\Pi\text{-}\mathsf{Decompositions}$

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quasi-bounded size	$n^{o(1)}$	nowhere dense [NO '11]	monadically stable [BNOS '25]

#### Theorem

A hereditary class  $\mathscr{C}$  of graphs is stable if and only if it admits quasibounded-size bounded shrubdepth decompositions.

# Sketch of the proof





Quasibounded-size bounded shurbdepth decompositions  $\Rightarrow$  stable:

- Quasibounded-size dependent decompositions dependent. Follows from existence of pre-coding (by an existential formula) of hereditary independent classes [BL '22+].
- Quasibounded-size stable decompositions ↔ stable.
  Follows from stable ↔ dependent and edge-stable [NOPRS '21].

### Sketch of the proof $\Leftarrow$

Quasibounded-size bounded shurbdepth decompositions  $\Leftarrow$  stable:

- Existence of distance-*r* neighborhood covers with almost nowhere dense incidence graph [DEMMPT '24]
- Improve the neighborhood cover so that distance between connected components in a same cluster is  $\geq 2r + 3$
- Flipper wins the flipper game of radius r in O(1) rounds [GMMOPPSST '23]
- Construct a decomposition tree by alternatively playing flipper game and constructing neighborhood covers
- Deduce a quasi-bush representation by quasi-bushes in an almost nowhere dense class
- Deduce the existence of quasibounded-size bounded shrubdepth decompositions [DGKPS '22].

### References

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# Some consequences



### Large homogeneous sets

#### Erdős-Hajnal property

Every graph in a hereditary stable class of graphs has a clique or an independent set of size  $\Omega(|G|^{1/2-\varepsilon})$  for every  $\varepsilon > 0$ .

#### Proof.

Consider quasibounded-size bounded shrubdepth decompositions with parameter 1.  $\hfill\square$ 

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#### Proof.

Consider quasibounded-size bounded shrubdepth decompositions with parameter 1.

#### Tightness

For every  $f : \mathbb{N} \to \mathbb{N}$ , there exists a hereditary stable class  $\mathscr{C}$  such that for every large c there exists  $G \in \mathscr{C}$  with

 $|G| \ge f(c)$  and  $\max(\alpha(G), \omega(G)) < |G|^{1/2-1/c}$ .

### Logarithmic density of induced subgraphs

#### Theorem [NO '11]

For every monotone nowhere dense class  $\mathscr{C}$  and every finite F we have

$$\limsup_{G \in \mathscr{C}} \frac{\log(\#F \subseteq G)}{\log|G|} \in \mathbb{N} \cup \{-\infty\}.$$

### Logarithmic density of induced subgraphs

#### Theorem

For every hereditary stable class  $\mathscr{C}$  and every finite F we have

$$\limsup_{G \in \mathscr{C}} \frac{\log(\#F \subseteq_i G)}{\log|G|} \in \mathbb{N} \cup \{-\infty\}.$$

# Thank you for your attention

