

Hereditary First-Order Logic and Extensional ESO

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Finite and Algorithmic Model Theory, Les Houches, 2025



European Research Council
Established by the European Commission

ERC Synergy Grant POCOCOP (GA 101071674)

Graph modification problems

Graph modification problems

Edge-modification

For a fixed graph class \mathcal{C} and an input graph G determine the minimum number of edge modifications to G so that it belongs to \mathcal{C} .

- ▶ Edge-deletion problems (Yanakkakis 1981).
- ▶ Hardness of Edge-Modification problems (Alon, Stav 2009).
- ▶ Dichotomy Results on the Hardness of H -free Edge Modification Problems (Aravind, Sandeep, Sivadasan 2017).
- ▶ Hardness of approximation for H -free edge modification problems (Bliznets, Cygan, Komosa, Pilipczuk 2018)

Graph modification problems

Resilience problems

For a fixed query μ , determine the resilience of μ in an input database \mathbb{D} .

- ▶ The Complexity of Resilience and Responsibility for Self-Join-Free Conjunctive Queries (Freire, Gatterbauer, Immerman, Meliou 2015)
- ▶ New Results for the Complexity of Resilience for Binary with Self-Joins (Freire, Gatterbauer, Immerman, Meliou 2020).
- ▶ A Unified Approach for Resilience and Causal Responsibility with Integer Linear Programming (ILP) and LP Relaxations. (Makhija, Gatterbauer 2023).
- ▶ The Complexity of Resilience Problems via Valued Constraint Satisfaction (Bodirsky, Semanišinová, Lutz 2024).

Graph modification problems

Vertex-deletion

For a fixed graph class \mathcal{C} and an input graph G determine the minimum k so that $G - U$ belongs to \mathcal{C} for some $|U| \leq k$.

- ▶ Node-Deletion NP-Complete Problems (Krishnamoorthy, Deo 1979)
- ▶ The node-deletion problem for hereditary properties is NP-complete (Lewis, Yanakkakis 1980).
- ▶ Finding odd-cycle transversals (Reed, Smith, Vetta 2004)
- ▶ On the Descriptive Complexity of Vertex Deletion Problems (Bannach, Chudigiewitsch, Tantau 2024)

Graph modification problems

Modification to first-order logic

Edge-modification: Given a graph G and a positive integer k test whether it is possible to modify at most k edges so that it satisfies ϕ .

Edge-completion: Given a graph G and a positive integer k test whether it is possible to add at most k edges from G so that it satisfies ϕ .

Edge-deletion: Given a graph G and a positive integer k test whether it is possible to remove at most k edges from G so that it satisfies ϕ .

Vertex-deletion: Given a graph G and a positive integer k test whether it is possible to remove at most k vertices from G so that it satisfies ϕ .

On the parameterized complexity of graph modification to first-order logic properties (Fomin, Golovach, Thilikos 2020)

Graph modification problems

Modification to first-order logic (without parameter k)

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Ex. 2 Chordal graphs: $\phi :=$ exists a simplicial vertex (Rose, 1970).

Ex. 3 Acyclic digraphs: $\phi :=$ exists a source.

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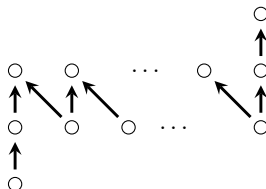
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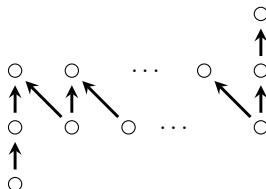
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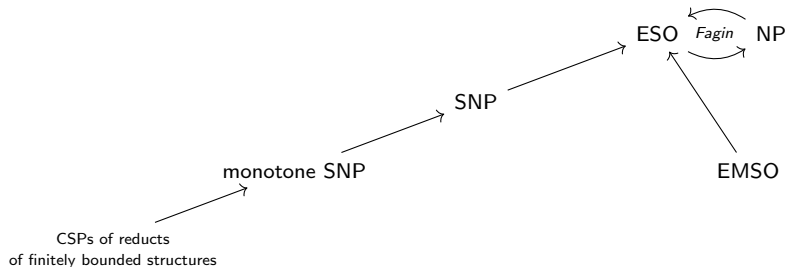
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Thm. If the quantifier prefix of ϕ is of the form $\forall^* \exists \forall^*$ or $\forall^* \exists^*$, then $\text{Her}(\phi)$ is in P and in SNP. For every other quantifier prefix Q , there is an FO formula ϕ with quantifier prefix Q such that $\text{Her}(\phi)$ is coNP-complete.

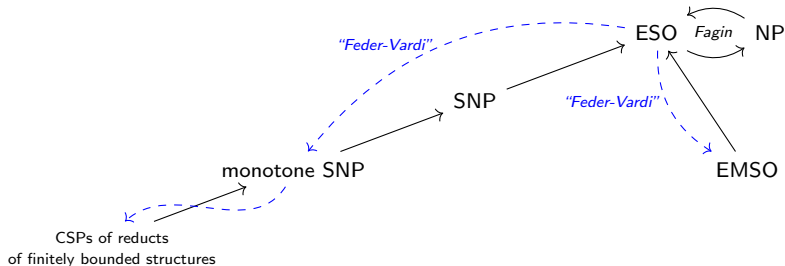
Hereditary first-order logic

“Qst.” What is the computational power of HerFO?

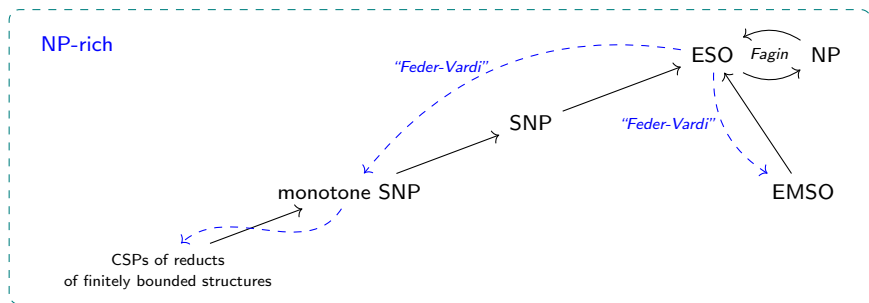
Existential Second-Order Logic



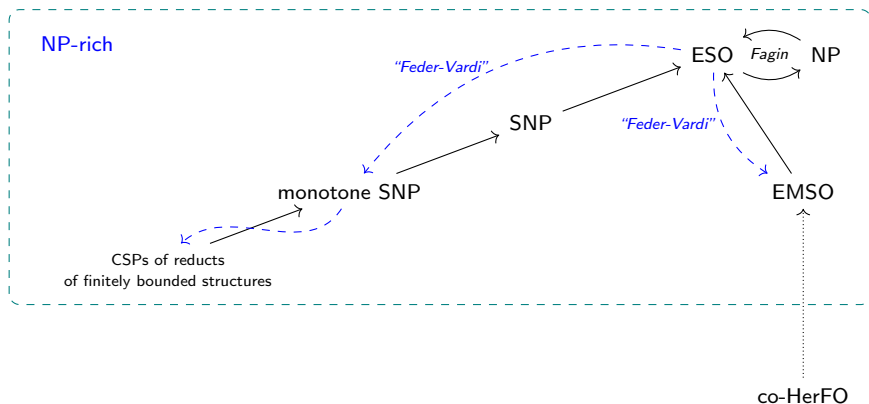
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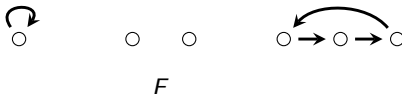


Extensional ESO

Extensional ESO

F-free edge-completion: Given a graph G test whether it is possible to add edges to G so that it becomes F -free.

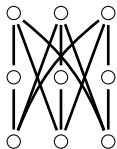
Ex 1. Acyclic digraphs: extend the edge relation to a (strict) linear order.



Extensional ESO

Edge-completion to ϕ : Given a graph G test whether it is possible to add edges so that it satisfies ϕ .

Ex 2. (Pach, 1971) A graph G has circular chromatic number < 3 iff G can be extended to a maximal triangle-free graphs that avoids:



Petersen minus vertex

Extensional ESO

Extensional ESO is the fragment of ESO

$$\exists R_1 \dots, R_k. \left[\bigwedge_{i \in [n]} \forall \bar{x} R'_i(x) \implies R(\bar{x}) \right] \wedge \phi$$

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Ex 3. $\exists W, B. \forall x. W'(x) \Rightarrow W(x) \wedge B'(x) \Rightarrow B(x) \wedge \phi.$

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Ex 4. A graph G is k -colourable iff it has a $\{K_1 + K_2, K_{k+1}\}$ -free extension.

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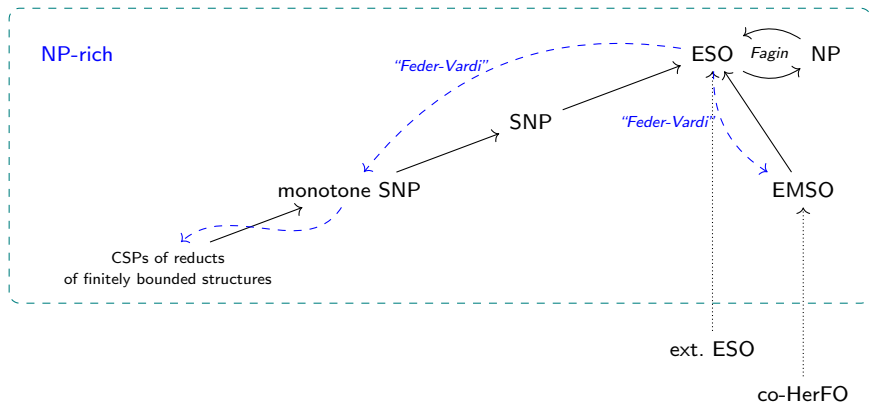
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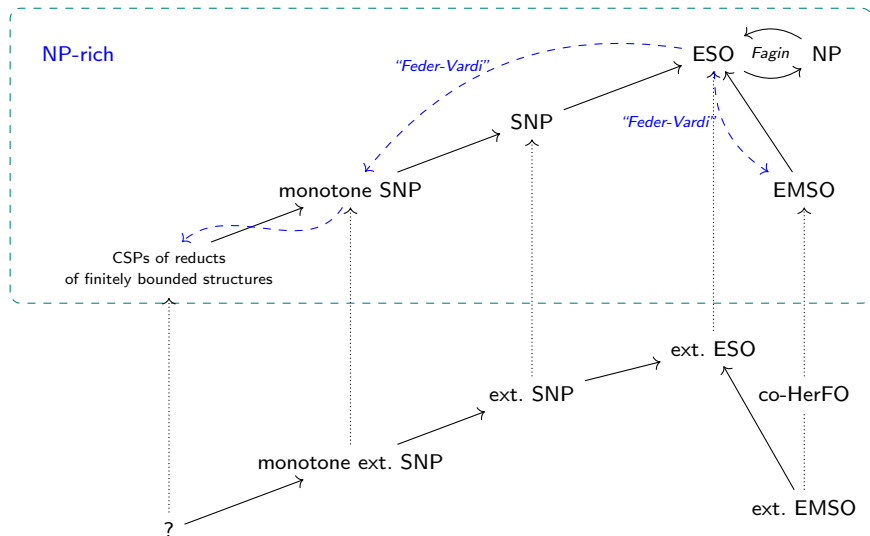
Ex 3. Pre-coloured $\text{CSP}(\mathbb{A})$ is in extensional ESO for every finite \mathbb{A} .

Ex 4. $\text{CSP}(\mathbb{A})$ and $\text{surj-CSP}(\mathbb{A})$ are in extensional ESO for every finite \mathbb{A} .

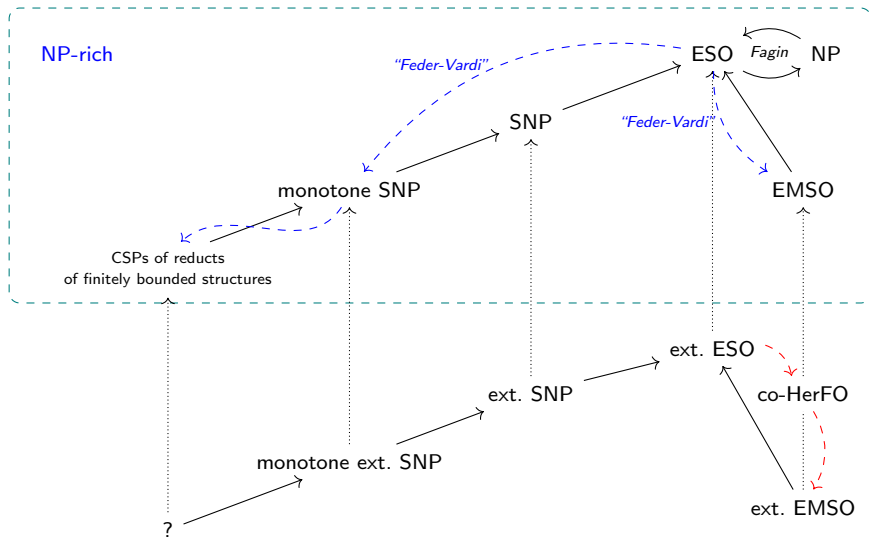
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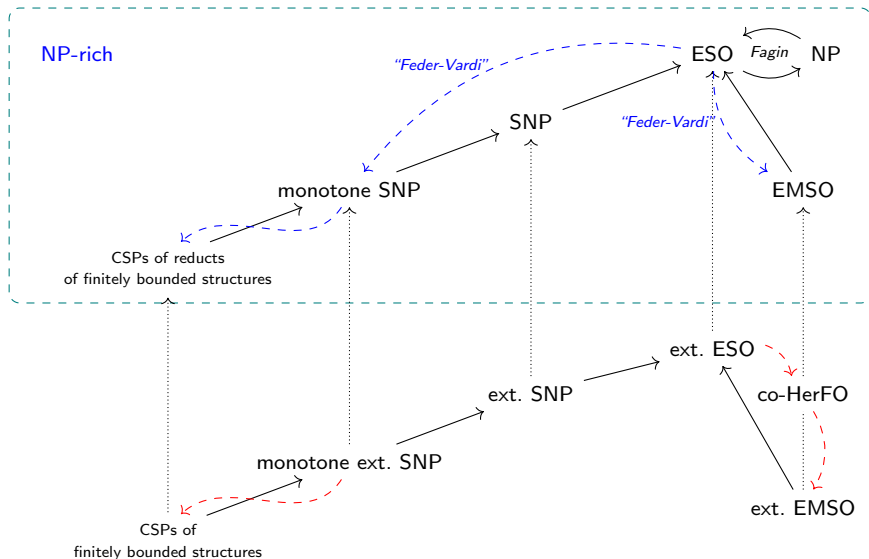
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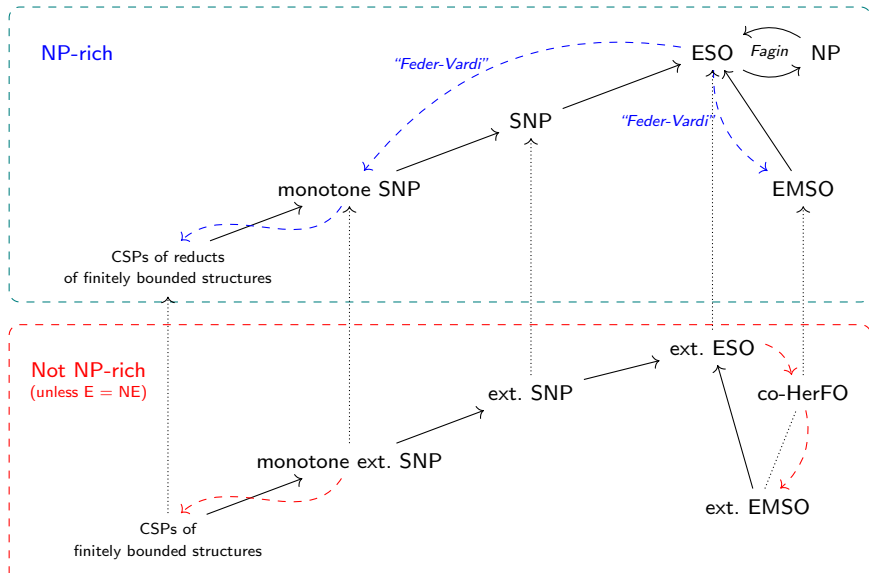
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Thank you for your attention!