

Semiring Semantics: Algebraic Properties vs. Logical Results

Sophie Brinke

Finite and Algorithmic Model Theory 2025, Les Houches

Semiring Semantics

Boolean semantics

τ -model \mathfrak{A} :

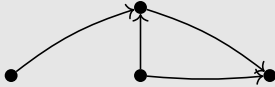


$\mathfrak{A} \models \exists x \exists y \exists z (Exy \wedge Eyz) =: \psi$

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add commutative semiring
 $(S, +, \cdot, 0, 1)$

Semiring semantics

What does the evaluation cost?

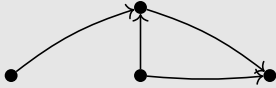
Which clearance level is required?

How many evaluation strategies are there?

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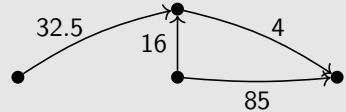
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$\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$

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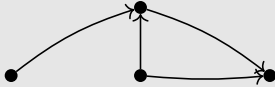
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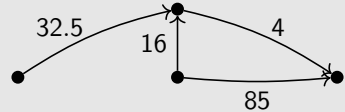
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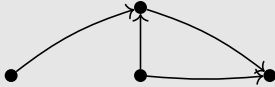


$$\pi \llbracket \psi \rrbracket = \min_{a,b,c} (\pi(Eab) + \pi(Ebc)) = 20$$

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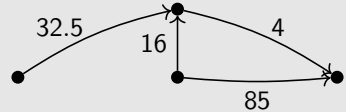
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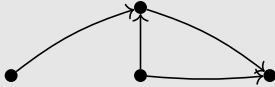


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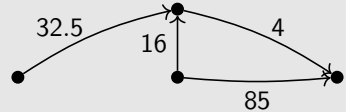
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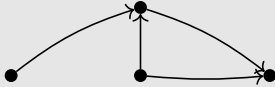


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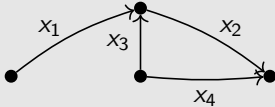
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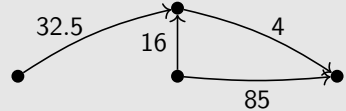
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$$\pi^*[\psi] = x_1 x_2 + x_3 x_2$$

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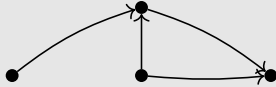


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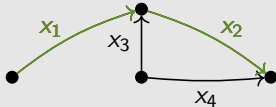
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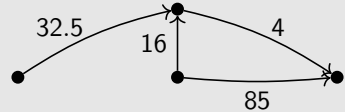
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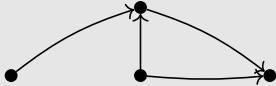


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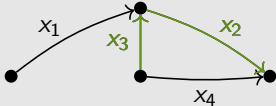
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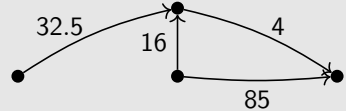
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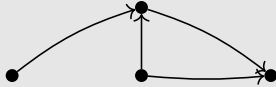


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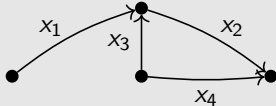
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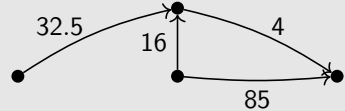
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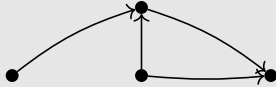


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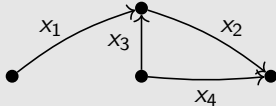
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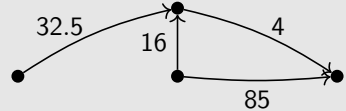
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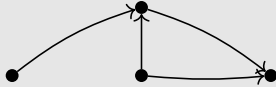


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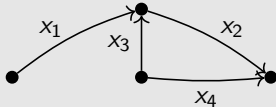
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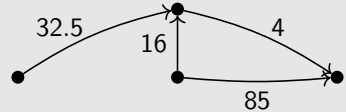
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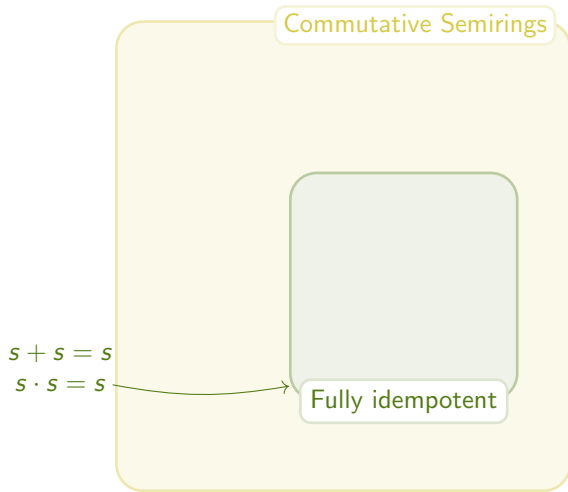
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Semirings and their Properties

Commutative Semirings



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$$s + s = s$$

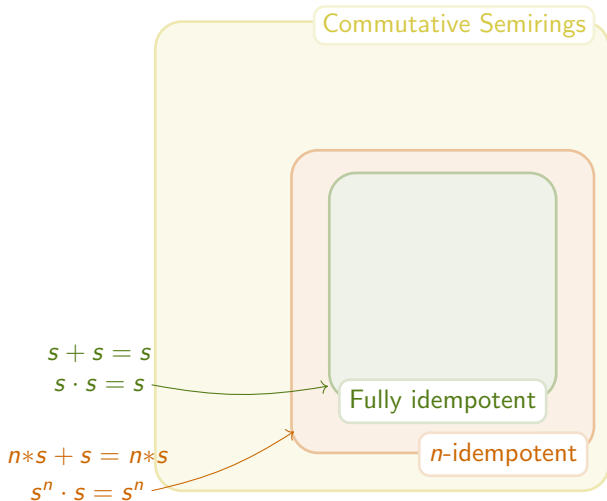
$$s \cdot s = s$$

Fully idempotent

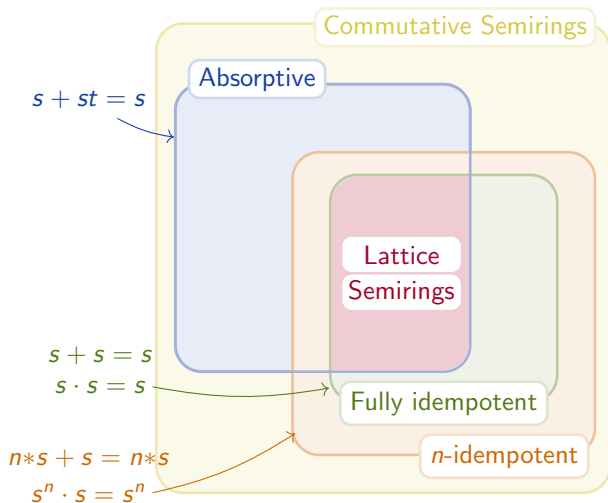
$$n*s + s = n*s$$

$$s^n \cdot s = s^n$$

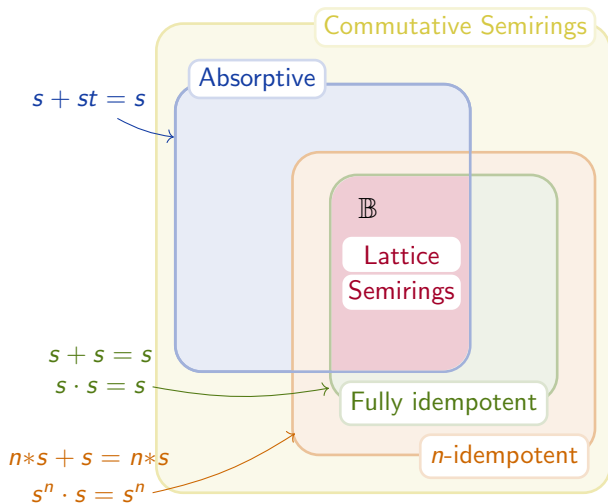
n -idempotent



Semirings and their Properties

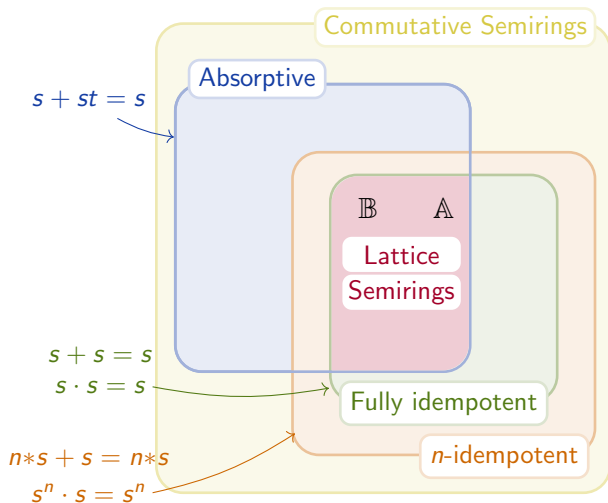


Semirings and their Properties



Classical semantics:
 $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$

Semirings and their Properties



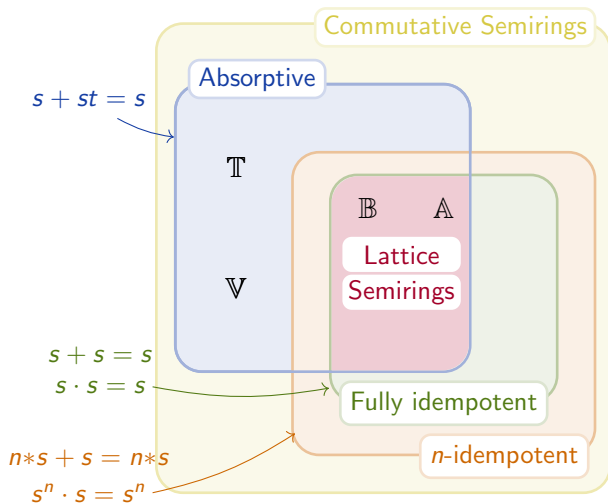
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Access restrictions:

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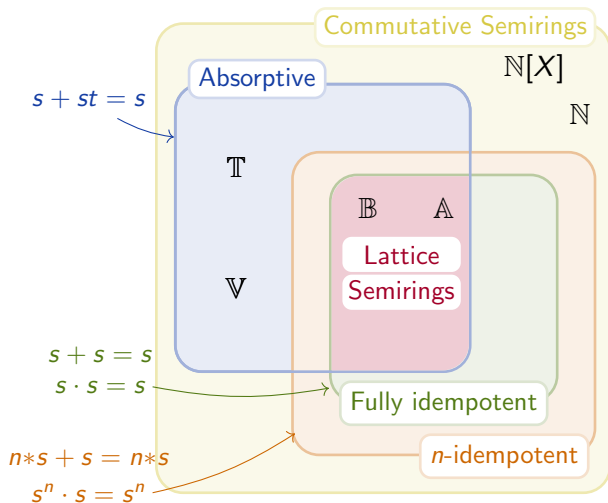
Cost analysis:

$$\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$$

Confidence:

$$\mathbb{V} = ([0, 1], \max, \cdot, 0, 1)$$

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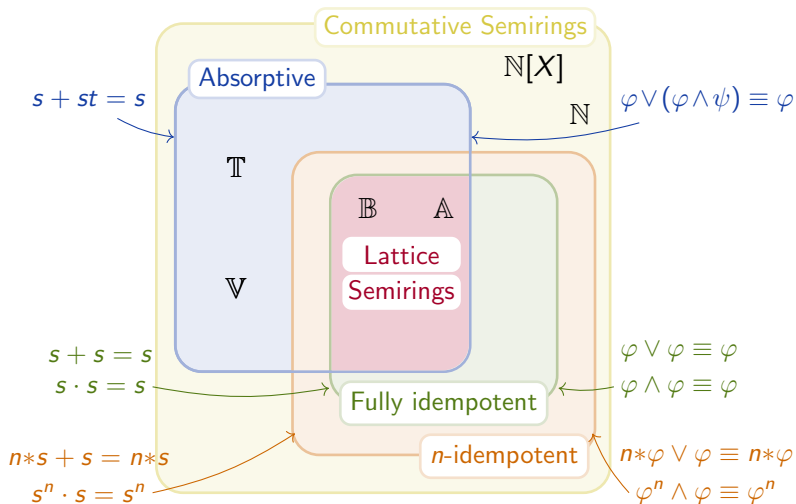
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...vs. Logical Results

Axiomatisability

$\mathfrak{A} \equiv \mathfrak{B} \Leftrightarrow \mathfrak{A} \cong \mathfrak{B}$
for finite $\mathfrak{A}, \mathfrak{B}$

Hanf Locality

Hanf equivalence
implies elementary
equivalence.

Compactness

$\Phi \models \psi \Leftrightarrow \Phi_0 \models \psi$
for some finite $\Phi_0 \subseteq \Phi$

EF Games

D wins $G_m(\mathfrak{A}, \mathfrak{B})$
 $\Leftrightarrow \mathfrak{A} \equiv_m \mathfrak{B}$

Gaifman Locality

Every $\varphi \in \text{FO}$ has a
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0–1 Laws

Every $\varphi \in \text{FO}$ almost
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\equiv vs. \cong in the Finite

$(\{0, 1, 2, 3\}, \max, \min, 0, 3)$

$\pi_A:$


A	P	Q	$\neg P$	$\neg Q$
a_1	1	3	0	0
a_2	2	1	0	0
a_3	3	2	0	0

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$\pi_A[\varphi] \stackrel{?}{=} \pi_B[\varphi]$ for all $\varphi \in \text{FO}$

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↑

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b_2	1	2	0	0
b_3	2	3	0	0

$\pi_A[\varphi] \stackrel{?}{=} \pi_B[\varphi]$ for all $\varphi \in \text{FO}$

How to prove
equivalence?

\equiv vs. \cong in the Finite

$(\{0, 1, 2, 3\}, \max, \min, 0, 3)$

$$\pi_A:$$

A	P	Q	$\neg P$	$\neg Q$
a_1	1	3	0	0
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$\stackrel{?}{\equiv}$

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\neq

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$$1 = \pi_A \llbracket \varphi \rrbracket \neq$$

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$$0 = h(1) = h(\pi_A[\![\varphi]\!]) \neq$$

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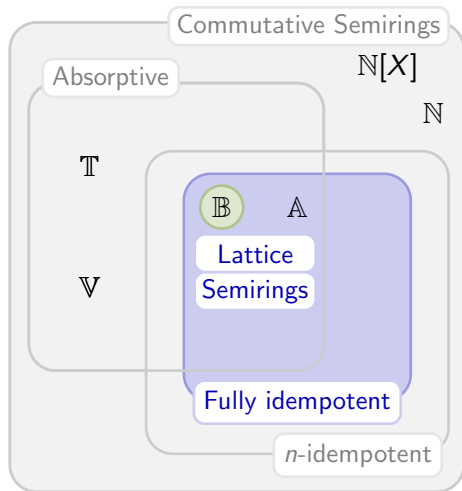
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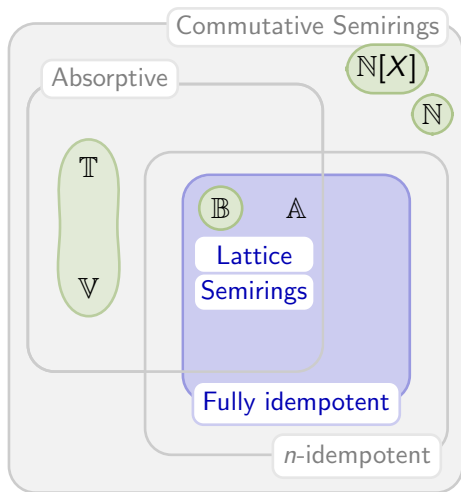
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Grädel, Mrkonjić, 2021

For any fully idempotent $\mathcal{S} \not\equiv \mathbb{B}$ there are finite \mathcal{S} -interpretations $\pi_A \not\equiv \pi_B$ such that $\pi_A \equiv \pi_B$.

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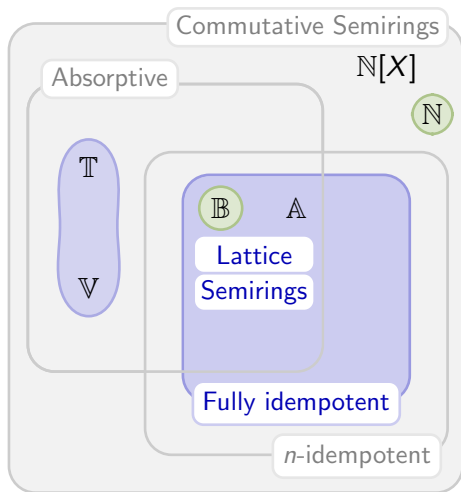


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Every finite \mathbb{N} -interpretation is axiomatisable by a single axiom. This is not true for \mathbb{V} and \mathbb{T} .

Logical Results

Axiomatisability

$\pi_A \equiv \pi_B \Leftrightarrow \pi_A \cong \pi_B$
for finite π_A, π_B

Hanf Locality

Hanf equivalence
implies elementary
equivalence.

Compactness

$\Phi \models \psi \Leftrightarrow \Phi_0 \models \psi$
for some finite $\Phi_0 \subseteq \Phi$

EF Games

D wins $G_m(\pi_A, \pi_B)$
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Gaifman Locality

Every $\varphi \in \text{FO}$ has a
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0-1 Laws

Every $\varphi \in \text{FO}$ almost
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 $s \in \mathcal{S}$

① How can classical model-theoretic notions be generalised?

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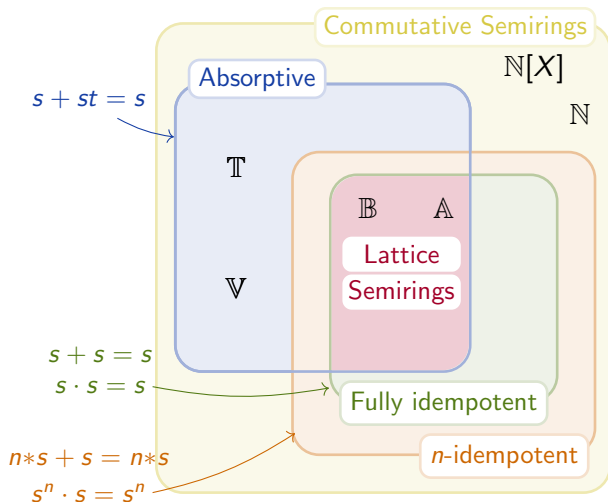
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Semirings and their Properties



Classical semantics:

$$\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$$

Access restrictions:

$$(S, \max, \min, s, t)$$

Cost analysis:

$$\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$$

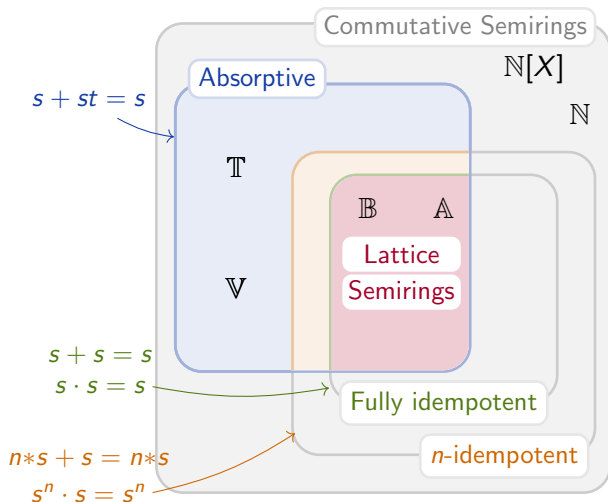
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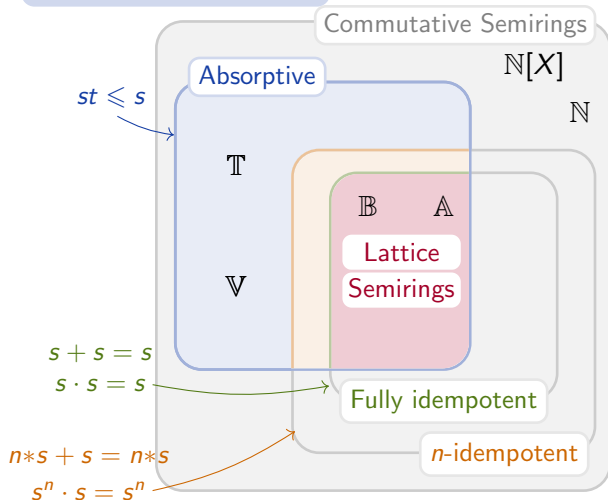
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$$u \leq v :\Leftrightarrow \exists w (u + w = v)$$



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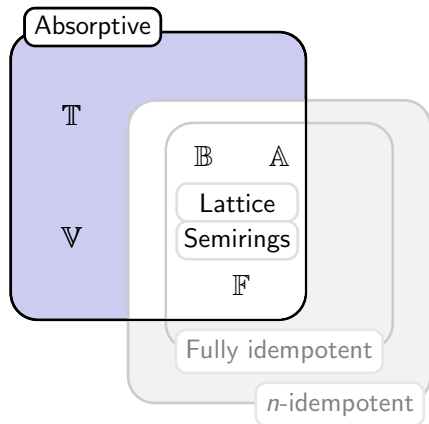
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Strong Compactness

B., Dawar, Grädel, Mrkonjić, Naaf, 2025

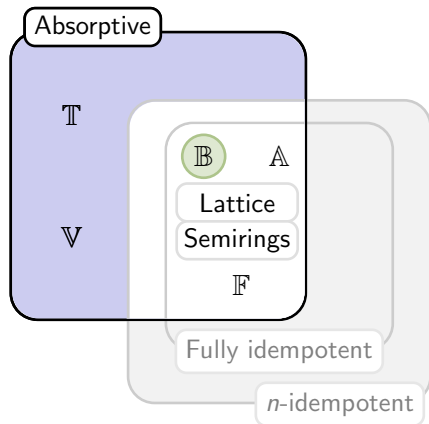
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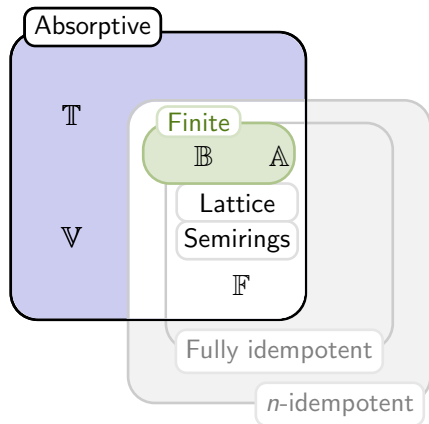


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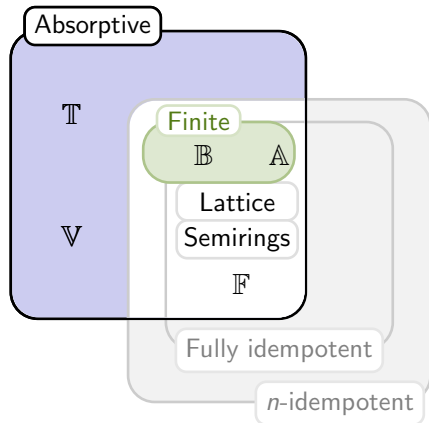
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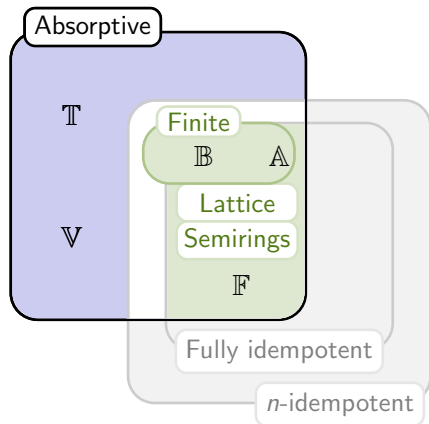
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→ Reduction to classical FO over signature $\{R_s, R_s^- \mid R \in \tau, s \in \mathcal{S}\}$



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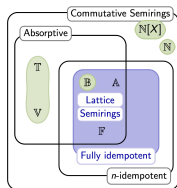
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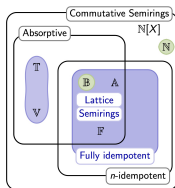
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Strong compactness also holds for lattice semirings.

Axiomatisability in the Finite

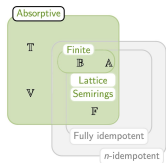


by $\Phi \subseteq FO$

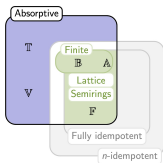


by $\varphi \in FO$

Compactness

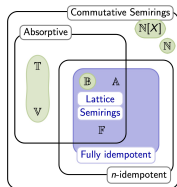


weak

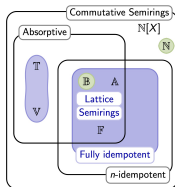


strong

Axiomatisability in the Finite

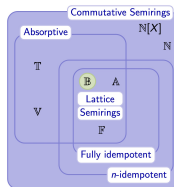


by $\Phi \subseteq \text{FO}$

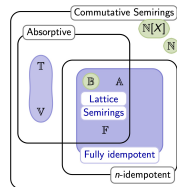


by $\varphi \in \text{FO}$

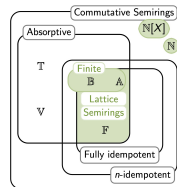
EF Games



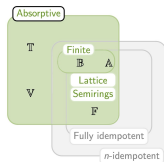
Bijection Games



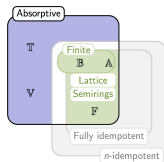
0-1 Laws



Compactness

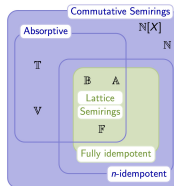


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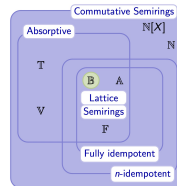


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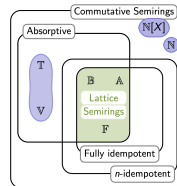
Hanf Locality



Gaifman Locality



for formulae



for sentences