# Succinct representations for join query evaluation

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Based on a joint work with Harry Vinall-Smeeth.

# Evaluating join queries using factorized representations

# Factorized databases Output [Olteanu, Závodný (TODS 2015)] Input evaluate $\varphi(x, y, z) = R(x, y) \land S(y) \land T(y, z)$

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### **Factorized databases** [Olteanu, Závodný (TODS 2015)] Input enumerate factorized representation count 34 evaluate $\varphi(x, y, z) =$ $R(x,y) \wedge S(y) \wedge T(y,z)$

- Compressed representation: much smaller than result relation.
- Efficient generation of result: enumeration with O(1) or delay.
- Counting (and other statistics) in O(poly(size of representation)).





- Inputs: partial mappings  $\{x \mapsto a\}$
- U-gates: union of partial mappings over same domains



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- for deterministic circuits of depth *d* and size *N*: *O*(*d*)-delay enumeration; *O*(*N*) counting



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• (deterministic)  $\{\cup, \times\}$ -circuits  $\approx$  multi-valued monotone (d-)DNNF

Another application of KC in DB theory: provenance polynomials (B Dan's talk)

• {U, ×}-circuits are strictly more succinct than multilinear & homogeneous provenance circuits over some semi-ring.



Recent surveys on this:

- Amarilli, Capelli: Tractable Circuits in Database Theory, 2024, http://arxiv.org/abs/2407.01127
- Amarilli, Arenas, Choi, Monet, Van Den Broeck, Wang: A Circus of Circuits: Connections Between Decision Diagrams, Circuits, and Automata, 2024, http://arxiv.org/abs/2404.09674

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 $\bigcup_{h:V(G)\to V(H)} \underset{x\in V(G)}{\times} \{x\mapsto h(x)\}$ 

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Theorem [Atserias, Grohe, Marx 2013]

The number of homomorphism from G to H is at most  $O(||H||^{\rho^*(G)})$ . For every G there are H such that there are  $\Omega(||H||^{\rho^*(G)})$  homomorphisms from G to H.

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- Worst case result relation:  $[N^{s_1}] \times \cdots \times [N^{s_k}]$  where  $(s_1, \ldots, s_k)$  max. fractional IS of G
- Has linear size circuits:  $X_{i=1}^k \bigcup_{a=1}^{s_i} \{x_i \mapsto a\}$

### For which queries can we always get small circuits?

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A tree shaped join query:



 $E(x_1, x_2) \land E(x_1, x_3) \land E(x_2, x_4) \land E(x_2, x_5) \land E(x_2, x_6) \land E(x_3, x_7) \land E(x_3, x_8)$ 

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#### Wires:

$$C_{v}^{i} = \left\{ x_{i} \mapsto v \right\} \times \bigotimes_{(x_{i}, x_{j}) \in E(G)} \bigcup_{(v, w) \in E(H)} C_{w}^{j}$$

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#### Output:

$$C = \bigcup_{v \in V(H)} C_v^1$$







**Main Idea:** For each bag  $\beta(t)$ ,  $t \in V(T)$  and partial homomorphism  $h: \beta(t) \to V(H)$ :  $C_h^t$  represents "subtree query" rooted at t and extending h.

#### Theorem [Olteanu, Závodný 2015]

For every G and arbitrarily large H there is a deterministic  $\{\cup, \times\}$ -circuit representing Hom(G, H) of size  $O(\|H\|^{tw(G)+1})$ .<sup>a</sup>

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Does not matter if union is disjoint in the worst-case.

Lower bound argument (1/3)

#### Lemma [B., Vinall-Smeeth 2023]

For any  $k \in \mathbb{N}$  there are arbitrary large graphs H with m edges such that any  $\{\cup, \times\}$ -circuit representing all k-cliques has size  $\Omega(m^{k/2}/\log^{3k-2}(m))$ .

• Flat representation has size  $O(\#k\text{-Cliques in }H) = O(m^{k/2})$ .

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For a lower bound we need instances with:

- many k-cliques
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We pick a random graph on *n* vertices (edge probability  $\frac{1}{2}$ ), w.h.p. it has  $\Omega(m^{k/2})$  k-cliques and no  $K_{a,a}$  subgraph for  $a \ge 3 \log n$ .

• A gate in the circuit is small if projection on each variable of computed relation is < 3 log n.

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  ⇒ It can only represent n log<sup>c</sup> n mappings.

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  - $\Rightarrow$  It can only represent  $n \log^c n$  mappings.
  - $\Rightarrow$  There are many such gates.

Lower bound argument (2/3)

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For every W there is a W-balanced separator of size tw(G) + 1:

Conversely, large treewidth  $\implies$  ex. large highly connected set:

#### Lemma

If tw(G) > 3k, then there is an  $W \subseteq V(G)$  of size 2k + 1 that has no W-balanced separator of size k.

• Each  $\{\cup, \times\}$ -circuit contains a gate g that defines a balanced partition  $(X, Y) = (\operatorname{var}(g), V \setminus \operatorname{var}(g))$  of W (where  $\frac{1}{3}|W| \le |X \cap W|, |Y \cap W| \le \frac{2}{3}|W|$ ).

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- Projection to at least one endpoint of each matching edge has to be small  $(< 3 \log n)$ .
- g responsible for few homomorphisms  $\Rightarrow$  there are many such g.

For which queries can we always get small circuits? **PART II:** Arbitrary relational structures

# Acyclic Join Queries

Relations of large arity: treewidth not the right measure

 $R(x_1, x_2, x_3, x_4, x_5) \land S(x_4, x_5, x_6, x_7) \land T(x_7, x_8, x_9)$ 



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- Acyclic query := has join-tree = tree decomp. where every bag corresponds to an atom.
- "projections on a bag" bounded by single relations in  ${\cal H}$
- get deterministic  $\{\cup, \times\}$ -circuits of size O(||H||)

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Lemma [Marx 2013] [B., Schweikardt 2019]

If subw(G)  $\leq w$ , then there are  $\ell = f(G)$  pairs  $(G_i, H_i)$ , where  $G_i$  is acyclic and  $||H_i|| \leq ||H||^{O(w)}$  such that

 $\operatorname{Hom}(G, H) = \operatorname{Hom}(G_1, H_1) \cup \cdots \cup \operatorname{Hom}(G_{\ell}, H_{\ell}).$ 

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- Union at top not disjoint  $\Rightarrow$  cannot be used for counting.
- Can be used for enumeration with  $f(G) ||H||^{O(subw(G))}$  preprocessing and f(G) delay. [B., Schweikardt 2019]

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For every G there are arbitrarily large H of size ||H|| such that any  $\{\cup, \times\}$ -circuit representing all homomorphisms from G to H has size

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Similar proof strategy as before:

- Use a "highly connected" set W (more complicated for subw than for tw). [Marx 2013]
- Define a worst-case instance with many homomorphisms.
- Put in randomness to destroy all large rectangles ("bipartite subgraphs"), while keeping many homomorphisms intact.
- Lower bound for rectangle cover again implies lower bound on circuit size.

Main results: bounds on the size of  $\{\cup, \times\}$ -circuits for homomorphisms from fixed G to large H:

• For graphs, digraphs:  $O(||H||^{tw(G)+1})$  and  $\Omega(||H||^{tw(G)/20})$ 

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**Open:** Characterisation for deterministic circuits in the general case. Main obstacles:

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Related CSP result for the other side [B., Mengel, Wilhelm 2024]:

If  ${\mathcal H}$  is a class of right-hand-side structures (databases), then

Hom(*G*, *H*) has polynomial-size  $\{\cup, \times\}$ -circuits for all *G* and  $H \in \mathcal{H}$ 

 $\ensuremath{\mathcal{H}}$  is strongly blockwise decomposable.