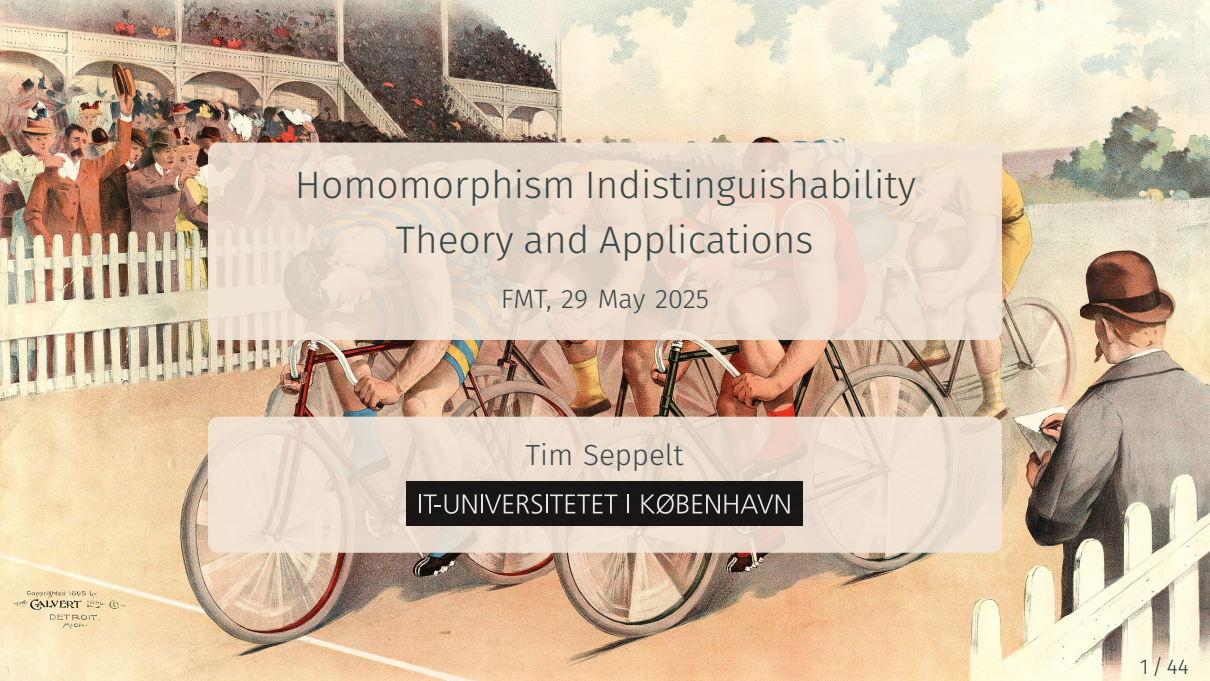


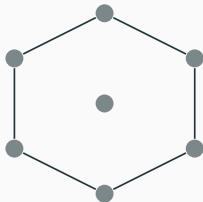
A vintage illustration of a bicycle race. In the foreground, several cyclists are racing on a dirt track. A jockey in a grey suit and brown bowler hat stands on the right, holding a clipboard. In the background, a large crowd of spectators in early 20th-century attire watches from a grandstand. The scene is set outdoors with a cloudy sky and some trees in the distance.

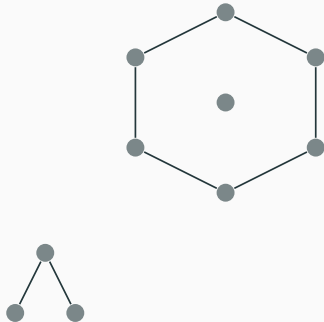
Homomorphism Indistinguishability Theory and Applications

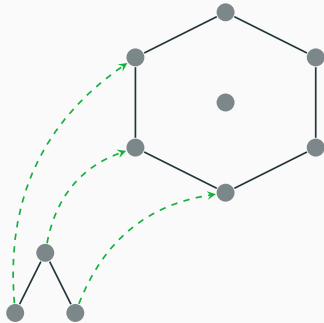
FMT, 29 May 2025

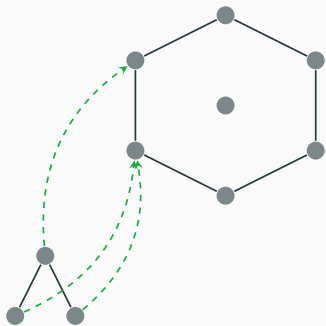
Tim Seppelt

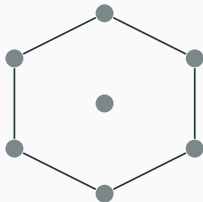
IT-UNIVERSITETET I KØBENHAVN



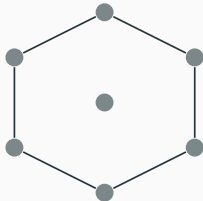








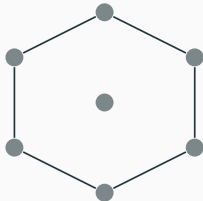
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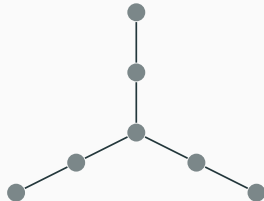
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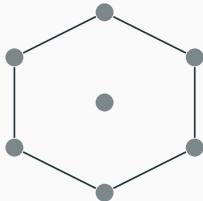


36



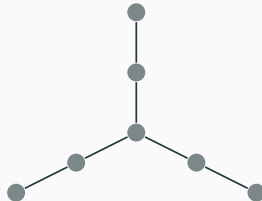
24

36



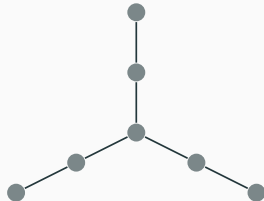
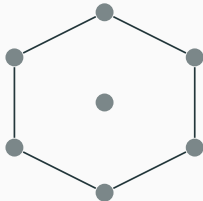
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24

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24



36

24

36

The graphs  and  are homomorphism indistinguishable over $\left\{ \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} , \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \right\}$.

graph class $\mathcal{F}$ relation $\equiv_{\mathcal{F}}$
all graphs isomorphism

Lovász (1967)

graph class $\mathcal{F}$	relation $\equiv_{\mathcal{F}}$
all graphs	isomorphism
cycles	cospectrality

Lovász (1967)

Folklore

graph class $\mathcal{F}$	relation $\equiv_{\mathcal{F}}$
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Lovász (1967)

Folklore

Mančinska & Roberson (2020)

graph class $\mathcal{F}$	relation $\equiv_{\mathcal{F}}$
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$\mathcal{TW}_k$	C^{k+1} -equivalence

Lovász (1967)

Folklore

Mančinska & Roberson (2020)

Dvořák (2010); Dell, Grohe, & Rattan (2018)

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	Graph Neural Networks	Xu, Hu, Leskovec, & Jegelka (2018); Morris, Ritzert, Fey, Hamilton, Lenssen, Rattan, & Grohe (2019)

$$\varphi(x) = \exists^{\leq t} y \ Exy$$



homomorphism
indistinguishability



$$XA_G = A_H X$$

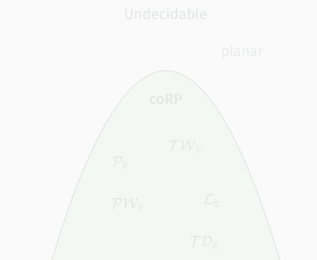
Characterisations

How to characterise $\equiv_{\mathcal{F}}$?



Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



Complexity

How to test $\equiv_{\mathcal{F}}$?

$$\varphi(x) = \exists^{=t}y \ Exy$$



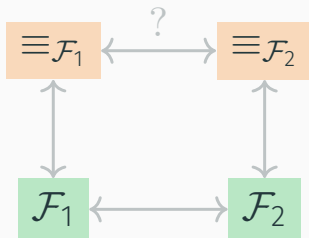
homomorphism
indistinguishability



$$XA_G = A_HX$$

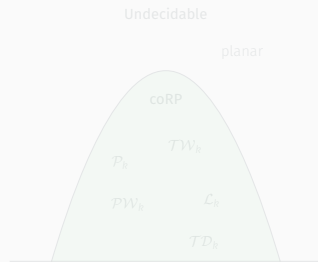
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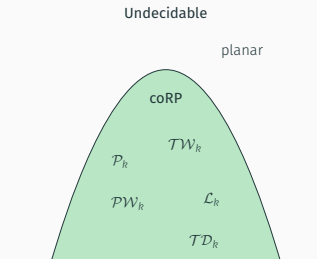
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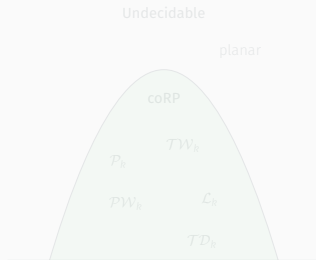
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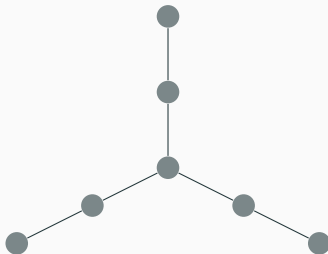
Observation

Two graphs *if, and only if, they are*
homomorphism indistinguishable over $\{\bullet\}$.

Observation

Two graphs
homomorphism indistinguishable over $\{\bullet\}$.

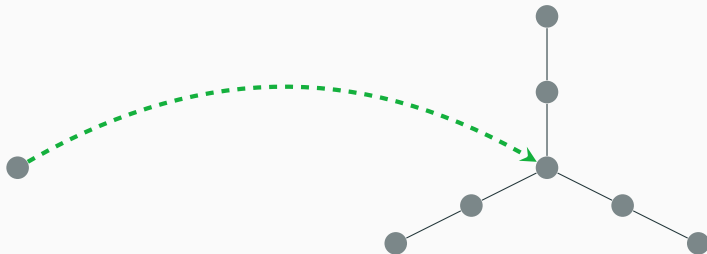
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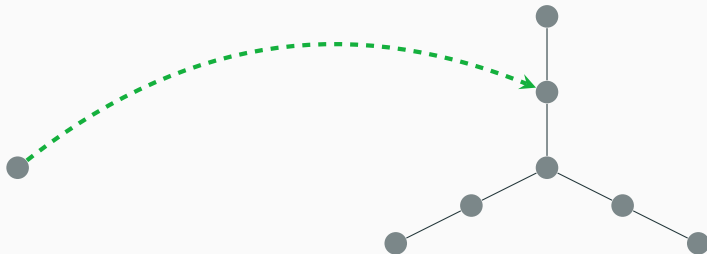
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Observation

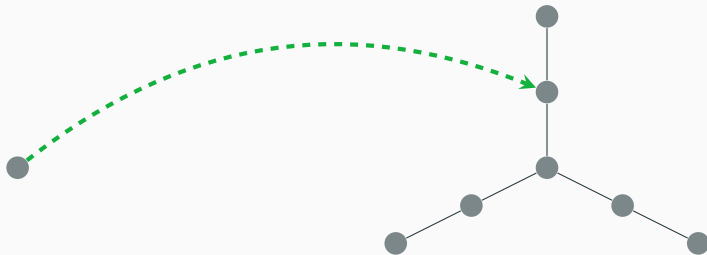
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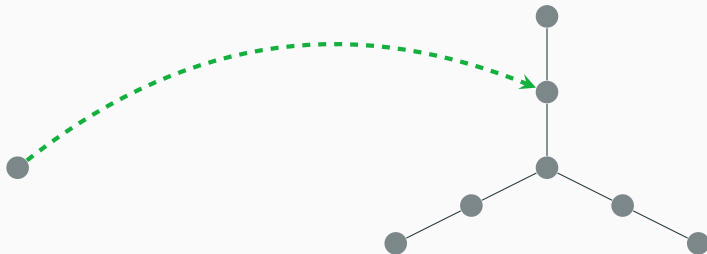
Two graphs *if, and only if, they are*
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$$\text{hom}(\bullet, G) = |V(G)|.$$

Observation

Two graphs *have the same number of vertices* if, and only if, they are homomorphism indistinguishable over $\{\bullet\}$.



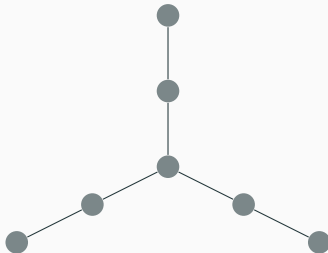
$$\text{hom}(\bullet, G) = |V(G)|.$$

Observation

Two graphs G and H are indistinguishable over $\{e\}$ if, and only if, they are homomorphic indistinguishable over $\{e\}$.

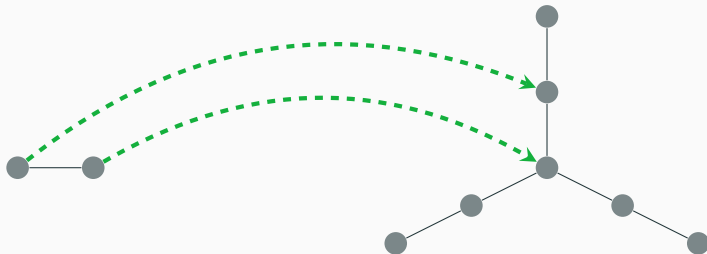
Observation

Two graphs G and H are indistinguishable by a graph homomorphism if, and only if, they are homomorphism indistinguishable over $\{\bullet-\bullet\}$.



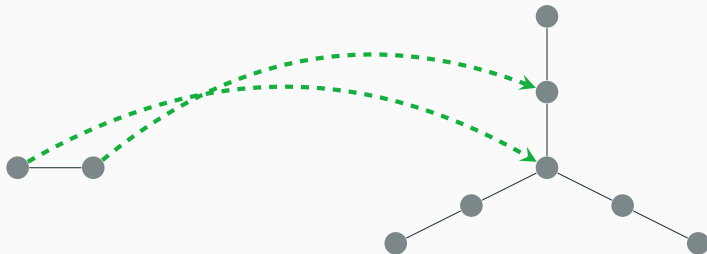
Observation

Two graphs
homomorphism indistinguishable over $\{\bullet-\bullet\}$.
if, and only if, they are



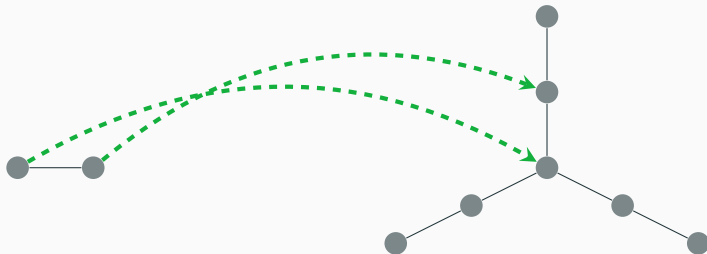
Observation

Two graphs *if, and only if, they are*
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Observation

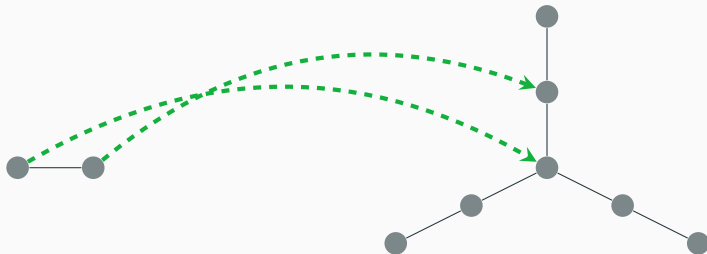
Two graphs G and H are indistinguishable by a graph homomorphism if, and only if, they are homomorphism indistinguishable over $\{\bullet-\bullet\}$.



$$\text{hom}(\bullet-\bullet, G) = 2|E(G)|.$$

Observation

Two graphs *have the same number of edges* if, and only if, they are homomorphism indistinguishable over $\{\bullet-\bullet\}$.



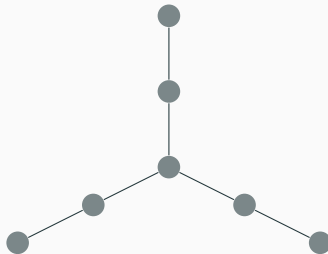
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Observation

Two graphs if, and only if, they are homomorphism indistinguishable over stars.

Observation

Two graphs *if, and only if, they are*
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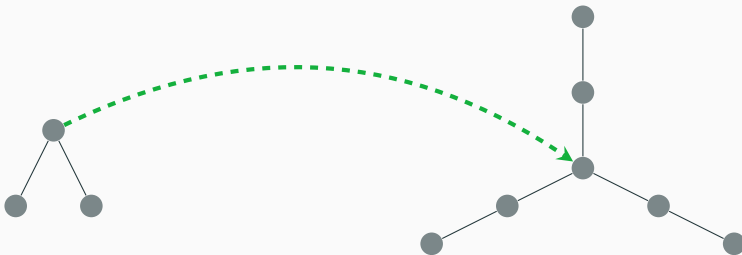


Observation

Two graphs

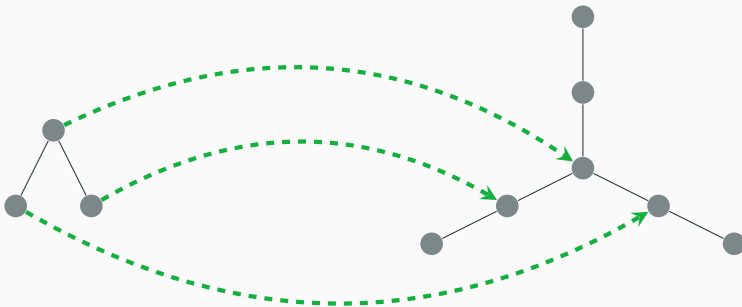
homomorphism indistinguishable over *stars*.

if, and only if, they are



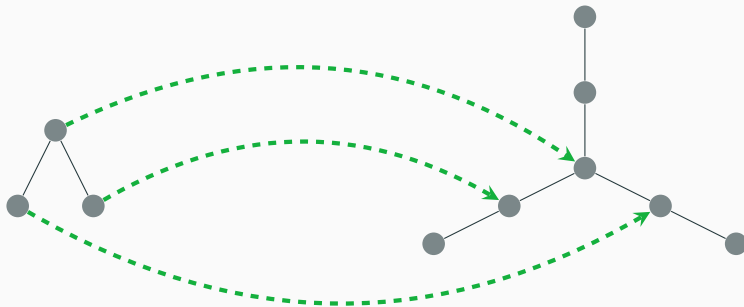
Observation

Two graphs *if, and only if, they are*
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Observation

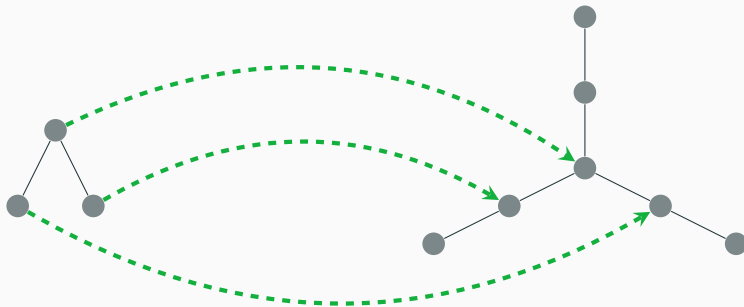
Two graphs G and H are ℓ -homomorphic if, and only if, they are indistinguishable over stars S_ℓ .



$$\text{hom}(S_\ell, G) = \sum_{v \in V(G)} (\deg(v))^\ell$$

Observation

Two graphs *have the same degree sequence* if, and only if, they are homomorphism indistinguishable over stars.



$$\text{hom}(S_\ell, G) = \sum_{v \in V(G)} (\deg(v))^\ell$$

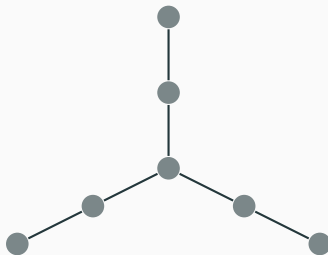
Theorem (Dvořák (2010); Dell, Grohe, & Rattan (2018))

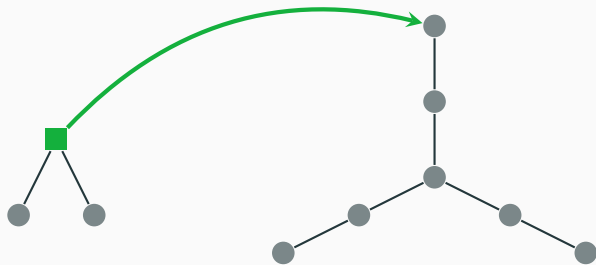
Two graphs are C^k -equivalent if, and only if, they are homomorphism indistinguishable over the graphs of treewidth $< k$.

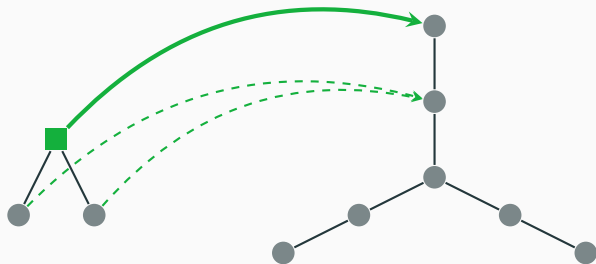
Theorem (Mančinska & Roberson (2020))

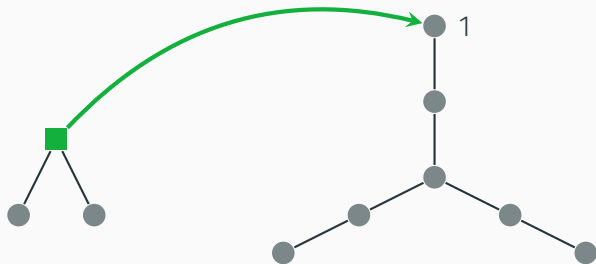
Two graphs are quantum isomorphic if, and only if, they are homomorphism indistinguishable over all planar graphs.

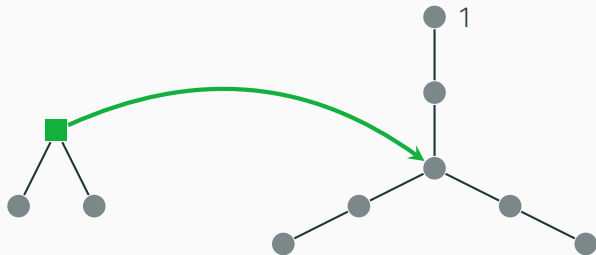


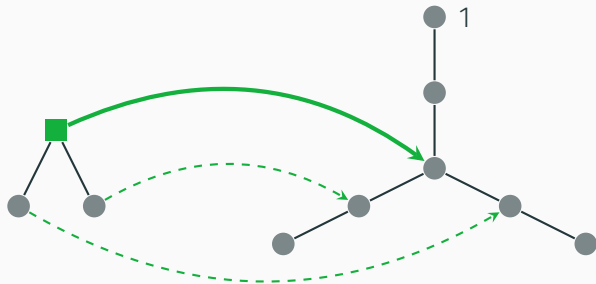


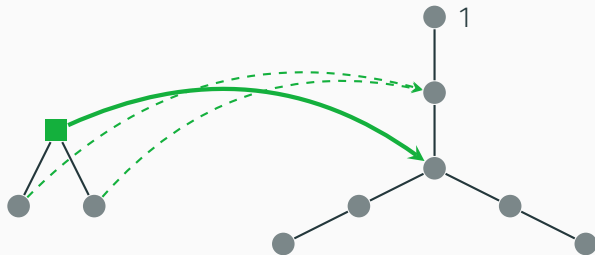


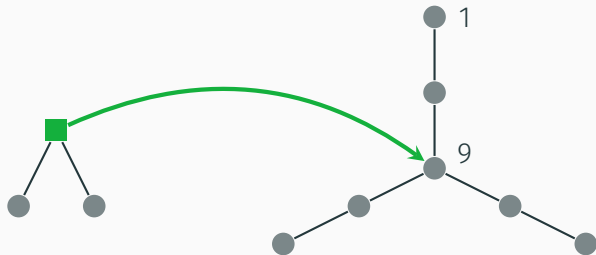




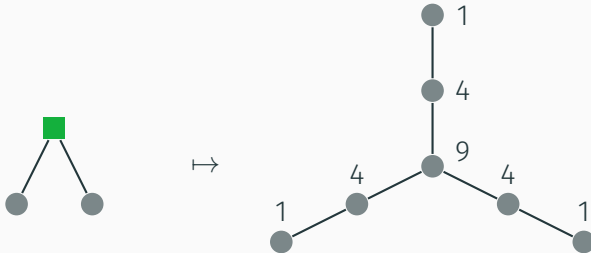






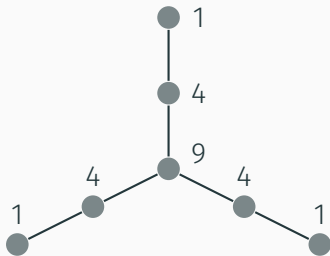


labelled graphs $\rightarrow$ homomorphism vectors $\subseteq \mathbb{R}^{V(G)}$



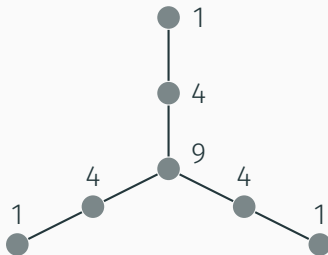


$\mapsto$





$\mapsto$

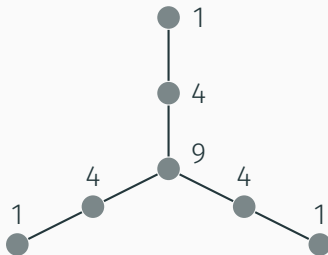


unlabelling $\Downarrow$





$\mapsto$



unlabelling $\Downarrow$

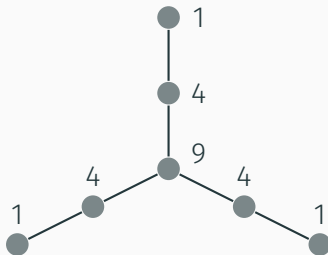


$\mapsto$

24



$\mapsto$



unlabelling $\Downarrow$



$\mapsto$

$\Downarrow$ sum of entries

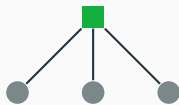
24

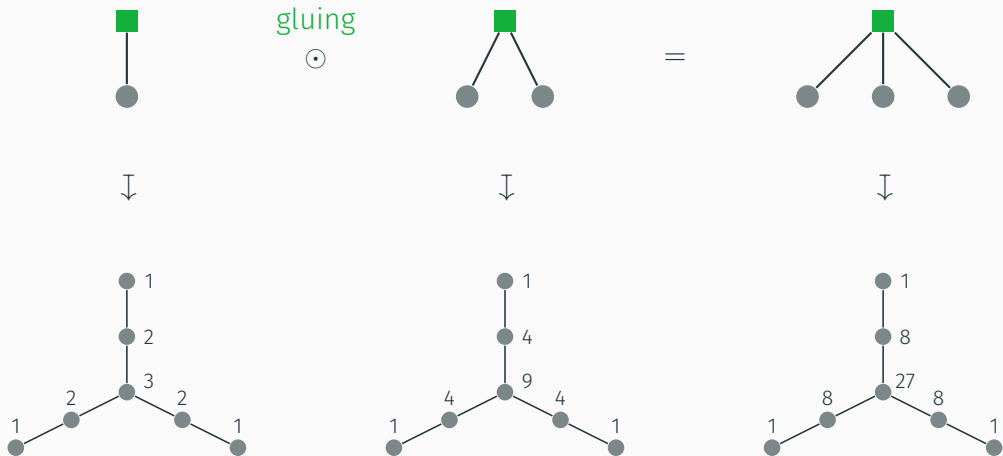


gluing
 $\odot$



=



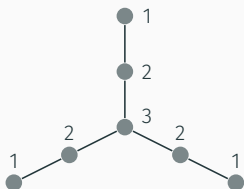




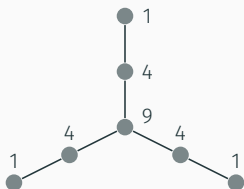
gluing



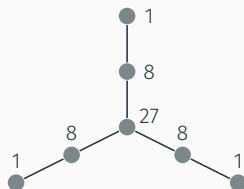
=



Schur
product



=



Theorem (Dvořák (2010); Dell, Grohe, & Rattan (2018))

Two graphs are C^k -equivalent if, and only if, they are homomorphism indistinguishable over the graphs of treewidth $< k$.

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C-formulas can be translated to (linear combinations of) homomorphism vectors and vice versa.

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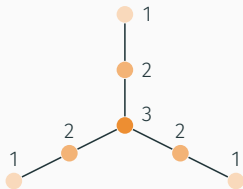
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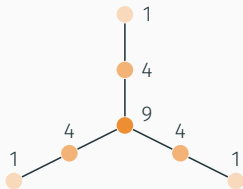
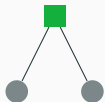


$$\varphi_t(x) = \exists^=t y \ Exy$$

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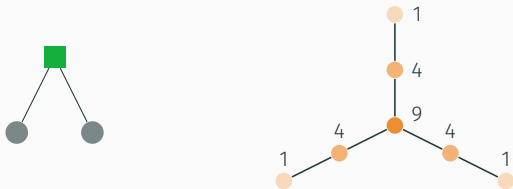


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$$\psi_t(x) = \bigvee_{t_1 t_2 = t} (\varphi_{t_1}(x) \wedge \varphi_{t_2}(x))$$

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Two graphs are C^k -equivalent if, and only if, they are homomorphism indistinguishable over the graphs of treewidth $< k$.

C_q treedepth $\leq q$

Grohe (2020)

C_q^k k -pebble forest cover of depth $\leq q$

Dawar, Jakl, & Reggio (2021)

Fluck, S., Spitzer (2024)

$\wedge C^k$ pathwidth $< k$

Montacute & Shah (2022)

$$\varphi(x) = \exists^{\neq t} y \ Exy$$



homomorphism
indistinguishability



$$XA_G = A_H X$$

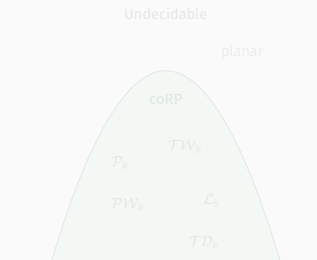
Characterisations

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Theorem (Mančinska & Roberson (2020))

Two graphs are *quantum isomorphic* if, and only if, they are homomorphism indistinguishable over all *planar graphs*.

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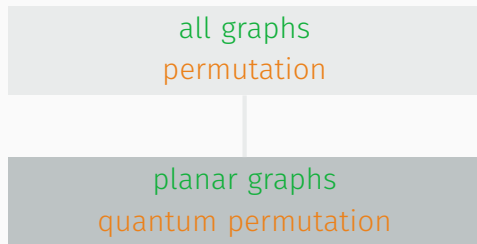
A matrix $X = (x_{ij})$ over some C^* -algebra is a *quantum permutation matrix* if

$$x_{ij}^2 = x_{ij} = x_{ij}^*, \quad \sum_k x_{ik} = 1 = \sum_k x_{kj}.$$

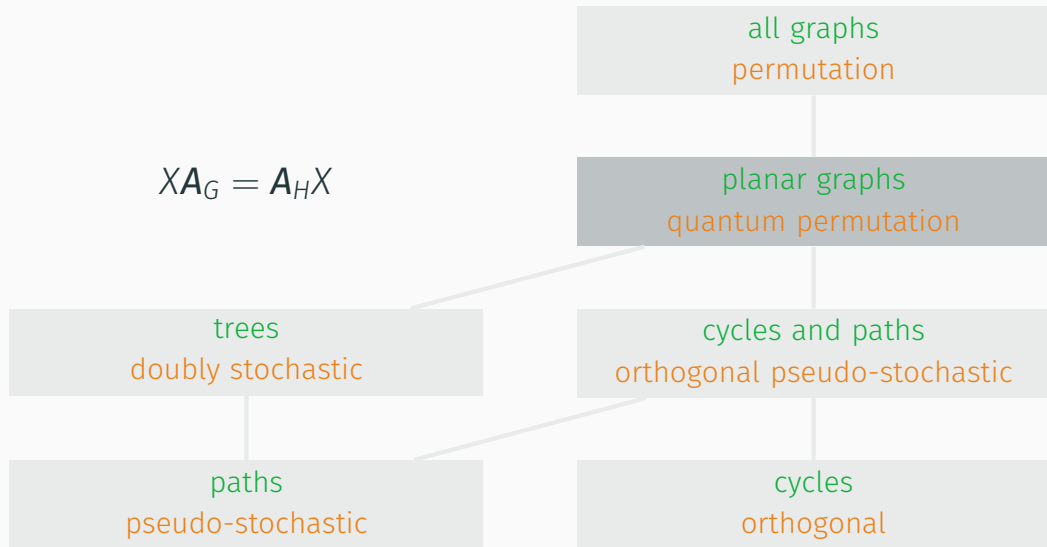
$$XA_G = A_HX$$

planar graphs
quantum permutation

$$XA_G = A_HX$$



$$XA_G = A_HX$$



Lupini, Mančinska, & Roberson (2020); Mančinska & Roberson (2020); Lovász (1967); Dell, Grohe, & Rattan (2018); Grohe, Rattan, & S. (2022); S. (2024).

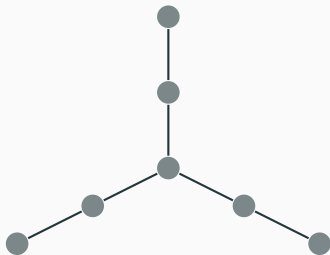


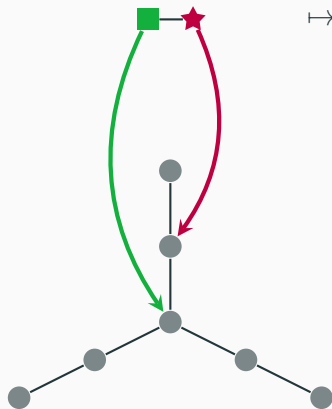
 $\mapsto$

}							
0	1	0	1	0	1	0	
1	0	1	0	0	0	0	
0	1	0	0	0	0	0	
1	0	0	0	1	0	0	
0	0	0	1	0	0	0	
1	0	0	0	0	0	1	
0	0	0	0	0	1	0	



$\mapsto$

$$\begin{Bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{Bmatrix}$$




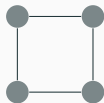
$\mapsto$

0	1	0	1	0	1	0
1	0	1	0	0	0	0
0	1	0	0	0	0	0
1	0	0	0	1	0	0
0	0	0	1	0	0	0
1	0	0	0	0	0	1
0	0	0	0	0	1	0





glue and
unlabel

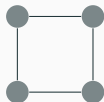




$\mapsto$

$$\left\{ \begin{array}{cccccc} 12 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 6 & 0 & 5 & 0 & 5 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 5 & 0 & 6 & 0 & 5 & 0 \\ 4 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 5 & 0 & 5 & 0 & 6 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 2 \end{array} \right.$$

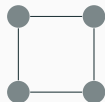
glue and
unlabel $\Downarrow$




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12	0	4	0	4	0	4
0	6	0	5	0	5	0
4	0	2	0	1	0	1
0	5	0	6	0	5	0
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36

Theorem

There is an *orthogonal* matrix X such that $XA_G = A_HX$ if, and only if, G and H are homomorphism indistinguishable over all *cycles*.

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Theorem (Specht (1940))

For symmetric matrices A and B , there is an *orthogonal* matrix X such that $XA = BX$ if, and only if, $\text{tr}(A^n) = \text{tr}(B^n)$ for all $n \in \mathbb{N}$.

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- Russell (2023): the NPA hierarchy converges.

$$\varphi(x) = \exists^{=t}y \ Exy$$



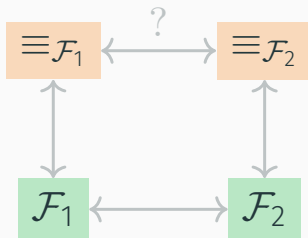
homomorphism
indistinguishability



$$XA_G = A_HX$$

Characterisations

How to characterise $\equiv_{\mathcal{F}}$?



Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



Complexity

How to test $\equiv_{\mathcal{F}}$?

Are all **cospectral** graphs isomorphic?

Günthard & Primas (1956)

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No!

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Homomorphism indistinguishability over **cycles** is not isomorphism.

Are all **C^k -equivalent** graphs isomorphic?

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Homomorphism indistinguishability over **graphs of treewidth $< k$** is not isomorphism.

Can $\equiv_{\mathcal{F}}$ be isomorphism for a proper graph class $\mathcal{F}$?

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Theorem (Dvořák (2010))

Two graphs are isomorphic if, and only if, they are homomorphism indistinguishable over all 2-degenerate graphs.

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Definition (Roberson (2022))

A graph class $\mathcal{F}$ is *homomorphism distinguishing closed* if, for all $F' \notin \mathcal{F}$,
there exist G and H such that $G \equiv_{\mathcal{F}} H$ and $\text{hom}(F', G) \neq \text{hom}(F', H)$.

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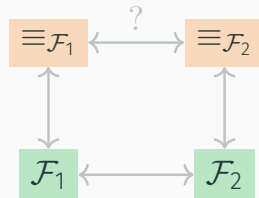
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Which *graph classes* are *homomorphism distinguishing closed*?

Observation

If $\mathcal{F}_1$ is homomorphism distinguishing closed, then

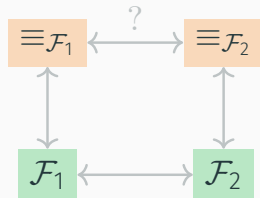
$$\equiv_{\mathcal{F}_1} \text{refines } \equiv_{\mathcal{F}_2} \iff \mathcal{F}_1 \text{ is a superclass of } \mathcal{F}_2.$$



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- optimisation
- machine learning
- finite model theory

Roberson & S. (2023)

Zhang, Gai, Du, Ye, He, & Wang (2024)

Adler, Fluck, S., & Spitzer (2025)

Which graph classes are homomorphism distinguishing closed?

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Conjecture (Roberson (2022))

Every **minor-closed union-closed** graph class is **homomorphism distinguishing closed**.

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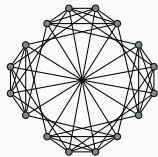
- planar graphs Roberson (2022)
- treewidth $\leq k$ Neuen (2024)
- treedepth $\leq q$ Fluck, S., & Spitzer (2024)
- k -pebble forest cover of depth $\leq q$ Adler & Fluck (2024)
- pathwidth $\leq k$ S. (2024)
- essentially finite graph classes S. (2023)
- outerplanar graphs Neuen & S. (2024)



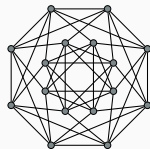
$$G \notin \mathcal{F}$$



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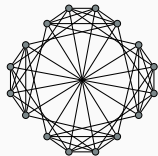
even CFI graph of G



odd CFI graph of G

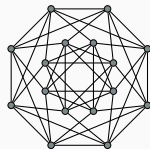


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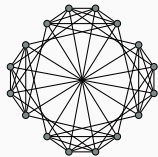
$G_0 \equiv_{\mathcal{F}} G_1$
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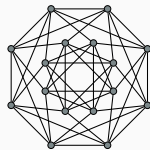
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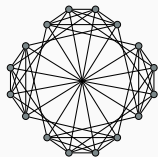
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Graph search-
ing game

Robber

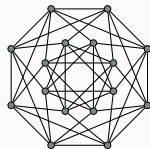


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Model com-
parison game

Duplicator

Theorem (Roberson (2022))

For a connected graph G and any graph F , tfae:

1. $\text{hom}(F, G_0) \neq \text{hom}(F, G_1)$,
2. there exists a *weak oddomorphism* $F \rightarrow G$.

Theorem (Roberson (2022))

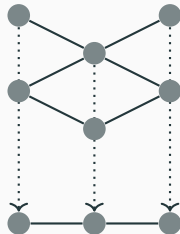
For a connected graph G and any graph F , tfae:

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If $F \rightarrow G$ is a **weak oddomorphism**, then

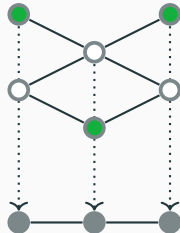
- $\text{tw}(F) \geq \text{tw}(G)$, Neuen (2024)
- $\text{td}(F) \geq \text{td}(G)$, Fluck, S., & Spitzer (2024)
- $\text{pw}(F) \geq \text{pw}(G)$, S. (2024)
- $F \text{ planar} \implies G \text{ planar}$, Roberson (2022)
- $\Delta(F) \geq \Delta(G)$, Roberson (2022)
- $F \text{ outerplanar} \implies G \text{ outerplanar}$. Neuen & S. (2024)

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A vertex $a \in V(F)$ is φ -even / φ -odd if $|N_F(a) \cap \varphi^{-1}(u)|$ is even / odd for every $u \in N_G(\varphi(a))$.

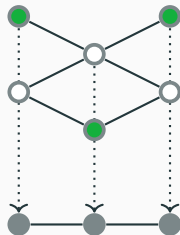


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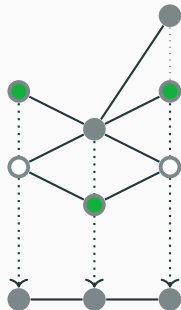
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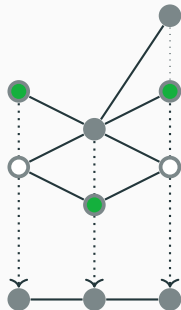
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Hypothesis

If $F \rightarrow G$ is a **weak odd morphism**, then G is a **minor** of F .

Theorem (S. (2023))

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$\mathcal{F}$ is minor-closed $\iff \equiv_{\mathcal{F}}$ is preserved under complements.

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- Typical graph isomorphism relaxations are preserved under complements.
- Towards a **theory of homomorphism indistinguishability**, we can focus on minor-closed graph classes.

Theorem (Roberson (2022))

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*There are uncountably many **homomorphism distinguishing closed** graph classes.*

Theorem (van Dobben de Bruyn, Marquès, Roberson, S., Zeman (2025+))

*There is a topology whose closed sets are precisely the **homomorphism distinguishing closed** sets.*

$$\varphi(x) = \exists^{\leq t} y \ Exy$$



homomorphism
indistinguishability



$$XA_G = A_HX$$

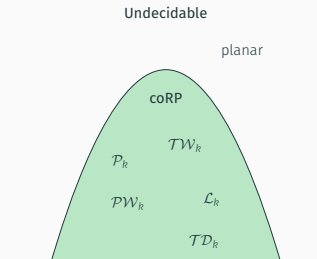
Characterisations

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Distinguishing Power

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Let $\mathcal{F}$ be minor-closed and proper.

HOMIND($\mathcal{F}$)

Input Graphs G and H .

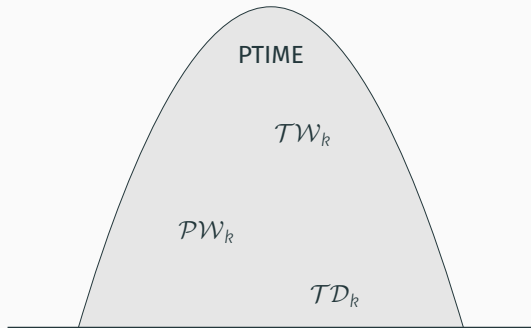
Decide $G \equiv_{\mathcal{F}} H$.

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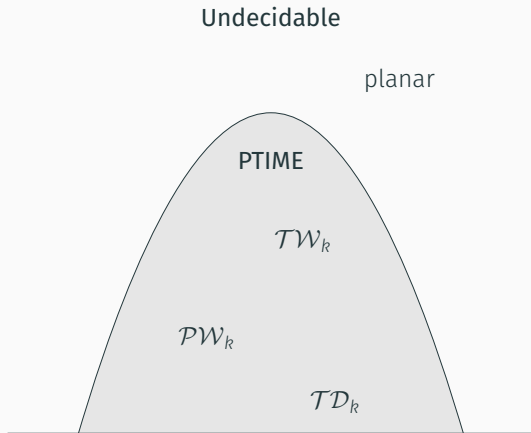
Dell, Grohe, & Rattan (2018); Dvořák (2010); Grohe (2020); Grohe, Rattan, S. (2022)

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Reduction to equivalence testing for $\mathbb{Q}$ -weighted tree automata, which is LOGSPACE interreducible with arithmetic circuit identity testing.

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Tzeng (1996)

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Corollary (S. (2024); Kar, Roberson, S., & Zeman (2025))

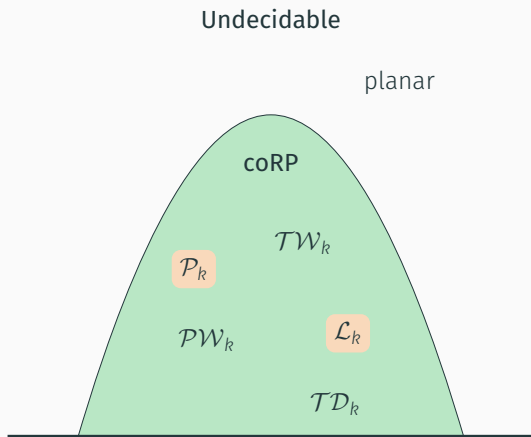
The following problems are in *coRP*:

- exact feasibility of the *level- k Lasserre relaxation of graph isomorphism*,
- exact feasibility of the *level- k NPA relaxation of quantum isomorphism*.

Let $\mathcal{F}$ be **minor-closed** and **proper**.

Theorem (S. (2024))

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Theorem (S. (2024))

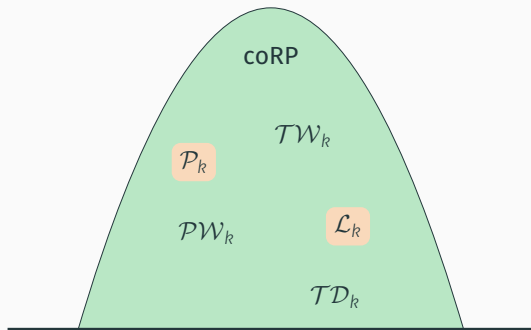
If $\mathcal{F}$ has **bounded treewidth**, then $\text{HOMIND}(\mathcal{F})$ is in **coRP**.

Conjecture (S. (2024))

If $\mathcal{F}$ has **bounded treewidth**, then $\text{HOMIND}(\mathcal{F})$ is in **PTIME**.

Undecidable

planar



Let $\mathcal{F}$ be **minor-closed** and **proper**.

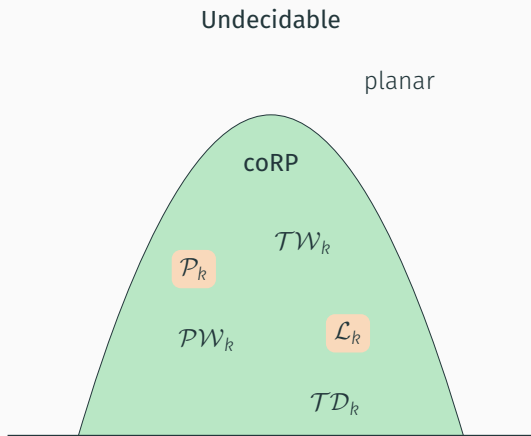
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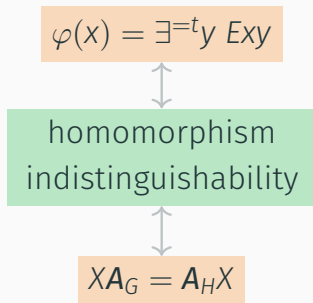
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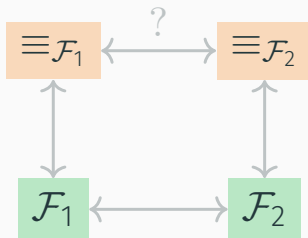
Otherwise, $\text{HOMIND}(\mathcal{F})$ is **undecidable**.





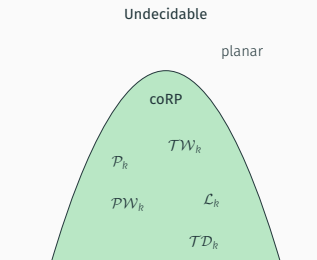
Characterisations

How to characterise $\equiv_{\mathcal{F}}$?



Distinguishing Power

What's the power of $\equiv_{\mathcal{F}}$?



Complexity

How to test $\equiv_{\mathcal{F}}$?

$$\varphi(x) = \exists^{=t}y \ Exy$$



homomorphism
indistinguishability



$$XA_G = A_HX$$

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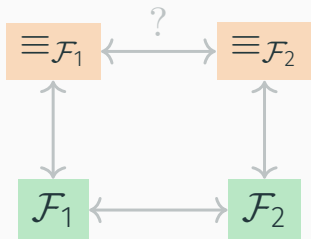


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Characterisations

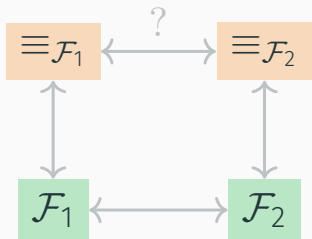
How to characterise $\equiv_{\mathcal{F}}$?

- Results by Lovász (1967); Dvořák (2010); Mančinska & Roberson (2020)
- **Tools:** labelled graphs and homomorphism vectors



Distinguishing Power

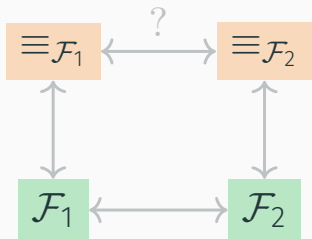
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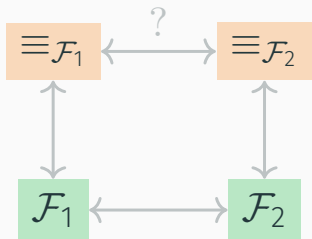
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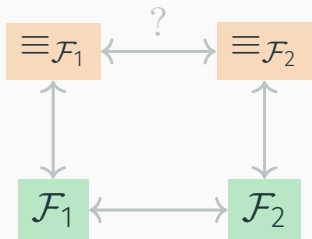
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- **minor-closed** graph classes play a central role.



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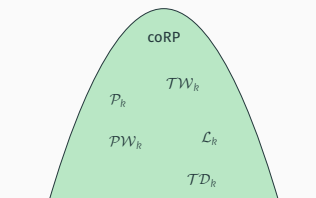
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Theory of Homomorphism Indistinguishability

- **minor-closed** graph classes play a central role.
- **Open:** Roberson's conjecture

Undecidable

planar

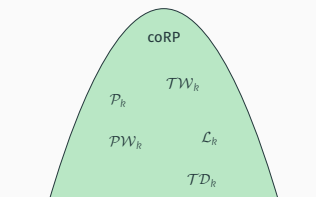


Complexity

How to test $\equiv_{\mathcal{F}}$?

Undecidable

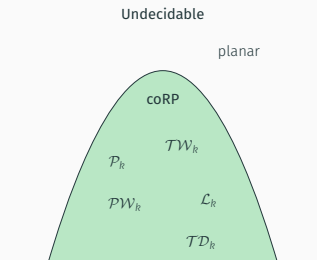
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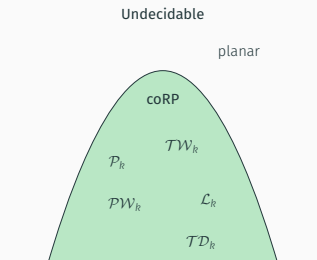


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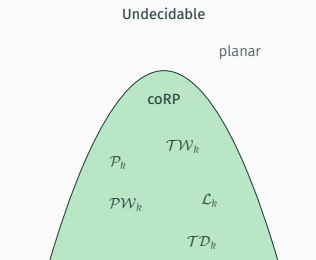


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- $\text{HOMIND}(\mathcal{F})$ is in **coRP** for **minor-closed** graph classes $\mathcal{F}$ of **bounded treewidth**.
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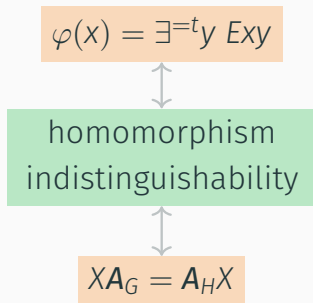
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- **coRP**-algorithms for SDP relaxations of (quantum) isomorphism



Homomorphism Indistinguishability Zoo

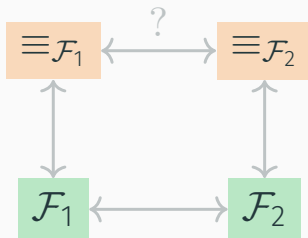
`tseppelt.github.io/homind-database`

Graph classes and their homomorphism indistinguishability properties.



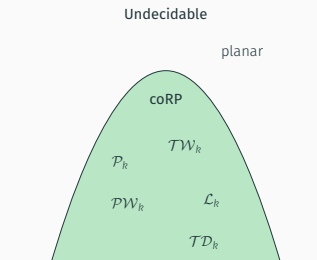
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-  Adler, Isolde & Eva Fluck (2024). 'Monotonicity of the Cops and Robber Game for Bounded Depth Treewidth'. In: *49th International Symposium on Mathematical Foundations of Computer Science (MFCS 2024)*. Ed. by Rastislav Kráľovič & Antonín Kučera. Vol. 306. Leibniz International Proceedings in Informatics (LIPIcs). ISSN: 1868-8969. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 6:1–6:18. ISBN: 978-3-95977-335-5. DOI: `10.4230/LIPIcs.MFCS.2024.6`. URL: `https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.MFCS.2024.6`.

-  Atserias, Albert & Elitza Maneva (2012). ‘Sherali–Adams Relaxations and Indistinguishability in Counting Logics’. In: *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*. ITCS ’12. Cambridge, Massachusetts: Association for Computing Machinery, pp. 367–379. ISBN: 9781450311151. DOI: 10.1145/2090236.2090265. URL: <https://doi.org/10.1145/2090236.2090265>.
-  Cai, Jin-Yi, Martin Fürer, & Neil Immerman (1992). ‘An optimal lower bound on the number of variables for graph identification’. In: *Combinatorica* 12.4, pp. 389–410. ISSN: 1439-6912. DOI: 10.1007/BF01305232. URL: <https://doi.org/10.1007/BF01305232>.

Bibliography iii

-  Collatz, Lothar & Ulrich Sinogowitz (Dec. 1957). 'Spektren endlicher Grafen. Wilhelm Blaschke zum 70. Geburtstag gewidmet'. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 21.1, pp. 63–77. DOI: 10.1007/BF02941924. URL: <http://link.springer.com/10.1007/BF02941924>.
-  Dawar, Anuj, Tomáš Jakl, & Luca Reggio (2021). 'Lovász-Type Theorems and Game Comonads'. In: *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*. IEEE, pp. 1–13. DOI: 10.1109/LICS52264.2021.9470609.
-  Dell, Holger, Martin Grohe, & Gaurav Rattan (2018). 'Lovász Meets Weisfeiler and Leman'. en. In: *45th International Colloquium on Automata, Languages, and Programming (ICALP 2018)*, 40:1–40:14. DOI: 10.4230/LIPICS.ICALP.2018.40.

-  Dvořák, Zdeněk (Aug. 2010). 'On recognizing graphs by numbers of homomorphisms'. en. In: *Journal of Graph Theory* 64.4, pp. 330–342. ISSN: 03649024. DOI: [10.1002/jgt.20461](https://doi.org/10.1002/jgt.20461). URL: <http://doi.wiley.com/10.1002/jgt.20461>.
-  Fluck, Eva, Tim Seppelt, & Gian Luca Spitzer (2024). 'Going Deep and Going Wide: Counting Logic and Homomorphism Indistinguishability over Graphs of Bounded Treedepth and Treewidth'. In: *32nd EACSL Annual Conference on Computer Science Logic (CSL 2024)*. Ed. by Aniello Murano & Alexandra Silva. Vol. 288. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 27:1–27:17. ISBN: 978-3-95977-310-2. DOI: [10.4230/LIPIcs.CSL.2024.27](https://doi.org/10.4230/LIPIcs.CSL.2024.27).

Bibliography v

-  Grohe, Martin (2020). 'Counting Bounded Tree Depth Homomorphisms'. In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. New York, NY, USA: Association for Computing Machinery, pp. 507–520. ISBN: 978-1-4503-7104-9. DOI: [10.1145/3373718.3394739](https://doi.org/10.1145/3373718.3394739).
-  Grohe, Martin & Martin Otto (2015). 'Pebble Games and Linear Equations'. In: *The Journal of Symbolic Logic* 80.3, pp. 797–844. ISSN: 00224812, 19435886. DOI: [10.1017/jsl.2015.28](https://doi.org/10.1017/jsl.2015.28). URL: <http://www.jstor.org/stable/43864249>.
-  Günthard, Hs. H. & H. Primas (1956). 'Zusammenhang von Graphentheorie und MO-Theorie von Molekeln mit Systemen konjugierter Bindungen'. In: *Helvetica Chimica Acta* 39.6, pp. 1645–1653. DOI: [10.1002/hlca.19560390623](https://doi.org/10.1002/hlca.19560390623). URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/hlca.19560390623>.

-  Kiefer, Stefan, Andrzej Murawski, Joel Ouaknine, Bjoern Wachter, & James Worrell (Mar. 2013). 'On the Complexity of Equivalence and Minimisation for Q-weighted Automata'. en. In: *Logical Methods in Computer Science* Volume 9, Issue 1, p. 908. ISSN: 1860-5974. DOI: [10.2168/LMCS-9\(1:8\)2013](https://doi.org/10.2168/LMCS-9(1:8)2013). URL: <https://lmcs.episciences.org/908>.
-  Lovász, László (Sept. 1967). 'Operations with structures'. In: *Acta Mathematica Academiae Scientiarum Hungarica* 18.3, pp. 321–328. ISSN: 1588-2632. DOI: [10.1007/BF02280291](https://doi.org/10.1007/BF02280291). URL: <https://doi.org/10.1007/BF02280291>.
-  Lupini, Martino, Laura Mančinska, & David E. Roberson (Sept. 2020). 'Nonlocal games and quantum permutation groups'. In: *Journal of Functional Analysis* 279.5, p. 108592. ISSN: 0022-1236. DOI: [10.1016/j.jfa.2020.108592](https://doi.org/10.1016/j.jfa.2020.108592). URL: <https://www.sciencedirect.com/science/article/pii/S002212362030135X>.

-  Malkin, Peter N. (May 2014). ‘Sherali–Adams relaxations of graph isomorphism polytopes’. In: *Discrete Optimization* 12, pp. 73–97. ISSN: 15725286. DOI: 10.1016/j.disopt.2014.01.004. URL: <https://linkinghub.elsevier.com/retrieve/pii/S157252861400005X>.
-  Mančinska, Laura & David E. Roberson (2020). ‘Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs’. In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pp. 661–672. DOI: 10.1109/FOCS46700.2020.00067.
-  Montacute, Yoàv & Nihil Shah (2022). ‘The Pebble-Relation Comonad in Finite Model Theory’. In: *LICS ’22: 37th Annual ACM/IEEE Symposium on Logic in Computer Science, Haifa, Israel, August 2 - 5, 2022*. Ed. by Christel Baier & Dana Fisman. ACM, 13:1–13:11. DOI: 10.1145/3531130.3533335. URL: <https://doi.org/10.1145/3531130.3533335>.

-  Morris, Christopher, Martin Ritzert, Matthias Fey, William L. Hamilton, Jan Eric Lenssen, Gaurav Rattan, & Martin Grohe (July 2019). 'Weisfeiler and Leman Go Neural: Higher-Order Graph Neural Networks'. In: *Proceedings of the AAAI Conference on Artificial Intelligence* 33, pp. 4602–4609. ISSN: 2374-3468, 2159-5399. DOI: [10.1609/aaai.v33i01.33014602](https://doi.org/10.1609/aaai.v33i01.33014602). URL: <https://aaai.org/ojs/index.php/AAAI/article/view/4384>.

-  Neuen, Daniel (2024). ‘Homomorphism-Distinguishing Closedness for Graphs of Bounded Tree-Width’. In: *41st International Symposium on Theoretical Aspects of Computer Science (STACS 2024)*. Ed. by Olaf Beyersdorff, Mamadou Moustapha Kanté, Orna Kupferman, & Daniel Lokshantov. Vol. 289. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 53:1–53:12. ISBN: 978-3-95977-311-9. DOI: [10.4230/LIPIcs.STACS.2024.53](https://doi.org/10.4230/LIPIcs.STACS.2024.53). URL: <https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.STACS.2024.53>.
-  Roberson, David E. (2022). *Oddomorphisms and homomorphism indistinguishability over graphs of bounded degree*. Number: arXiv:2206.10321. arXiv: 2206.10321[math]. URL: <http://arxiv.org/abs/2206.10321>.

-  Russell, Travis B. (2023). 'A synchronous NPA hierarchy with applications'. en. In: *Operators and Matrices* 4, pp. 901–924. ISSN: 1846-3886. DOI: 10.7153/oam-2023-17-60. URL: <https://oam.ele-math.com/17-60>.
-  Seppelt, Tim (2023). 'Logical Equivalences, Homomorphism Indistinguishability, and Forbidden Minors'. In: *48th International Symposium on Mathematical Foundations of Computer Science (MFCS 2023)*. Ed. by Jérôme Leroux, Sylvain Lombardy, & David Peleg. Vol. 272. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 82:1–82:15. ISBN: 978-3-95977-292-1. DOI: 10.4230/LIPIcs.MFCS.2023.82.

-  Specht, Wilhelm (1940). 'Zur Theorie der Matrizen. II.'. In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 50, pp. 19–23. ISSN: 0012-0456. URL: http://gdz.sub.uni-goettingen.de/dms/load/toc/?PPN=PPN37721857X_0050&DMDID=dmdlog6.
-  Tzeng, Wen-Guey (Apr. 1996). 'On path equivalence of nondeterministic finite automata'. en. In: *Information Processing Letters* 58.1, pp. 43–46. ISSN: 00200190. DOI: 10.1016/0020-0190(96)00039-7. URL: <https://linkinghub.elsevier.com/retrieve/pii/0020019096000397>.
-  Xu, Keyulu, Weihua Hu, Jure Leskovec, & Stefanie Jegelka (21st Dec. 2018). 'How Powerful are Graph Neural Networks?' In: *International Conference on Learning Representations*. URL: <https://openreview.net/forum?id=ryGs6iA5Km>.

 Zhang, Bohang, Jingchu Gai, Yiheng Du, Qiwei Ye, Di He, & Liwei Wang (2024). 'Beyond Weisfeiler–Lehman: A Quantitative Framework for GNN Expressiveness'. In: *The Twelfth International Conference on Learning Representations*. URL: <https://openreview.net/forum?id=HSKaG0i7Ar>.

Title Picture: 'Bicycle race scene. A peloton of six cyclists crosses the finish line in front of a crowded grandstand, observed by a referee.' (1895) by Calvert Lithographic Co., Detroit, Michigan, Public Domain, via Wikimedia Commons. https://commons.wikimedia.org/wiki/File:Bicycle_race_scene,_1895.jpg