Equi-rank Homomorphism Preservation Theorem on Finite Structures

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Łoś-Tarski / Lyndon / Homomorphism Preservation Theorems

A first-order sentence is

preserved under ***injective / surjective / all*** homomorphisms

if, and only if, it is

equivalent to an ***existential / positive / existential-positive*** sentence.

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equivalent to an ***existential / positive / existential-positive*** sentence.

- Originally proved in the 1950's using **compactness**.
- Proofs give **no computable upper bound** on the quantifier rank of the equivalent existential / positive / existential-positive sentence.
- Proofs relativize to any FO-axiomatizable class of structures.

  Proofs **do not relativize** to the non-FO-axiomatizable class of ***finite structures***.

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Question [Gurevich '84]

What is the status of classical theorems when restricted to ***finite structures***?

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Question [Gurevich '84]

What is the status of classical theorems when restricted to ***finite structures***?

- **Compactness** fails on finite structures.
- The **Los-Tarski** and **Lyndon Preservation Theorems** fail on finite structures [Tait '59, Ajtai-Gurevich '87].
- In contrast, the **HPT** is valid on finite structures [R.'05].

Equi-rank (Classical) HPT [R.'05]

Every hom-preserved sentence with quant. rank r
is equivalent to an existential-positive sentence with quant. rank r

HPT on Finite Structures [R.'05]

On finite structures, every hom-preserved sentence with quant. rank r
is equivalent to an existential-positive sentence with quant. rank $f(r)$
for some computable (but non-elementary) function $f : \mathbb{N} \to \mathbb{N}$

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- MAIN LEMMA

  Every structure $\mathfrak{A}$ has an (infinite) $\exists^+$-*saturated co-retract* $\widehat{\mathfrak{A}} \xrightarrow{\supseteq} \mathfrak{A}$ with the property that

$$\mathfrak{A} \equiv_r^{\exists^+} \mathfrak{B} \implies \widehat{\mathfrak{A}} \equiv_r^{\mathsf{FO}} \widehat{\mathfrak{B}}$$

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- MAIN LEMMA

  Every finite structure $\mathfrak{A}$ has an finite co-retract $\widehat{\mathfrak{A}}_r$, with the property that

$$\mathfrak{A} \equiv^{\exists^+}_{f(r)} \mathfrak{B} \implies \widehat{\mathfrak{A}}_r \equiv^{\mathsf{FO}}_r \widehat{\mathfrak{B}}_r$$

- REMARK

  The $\exists^+$-saturated co-retract $\widehat{\mathfrak{A}}$ (in the equi-rank proof) is a direct limit

$$\mathfrak{A} = \widehat{\mathfrak{A}}_0 \xleftarrow{\subseteq} \widehat{\mathfrak{A}}_1 \xleftarrow{\subseteq} \widehat{\mathfrak{A}}_2 \xleftarrow{\subseteq} \cdots \xleftarrow{\subseteq} \widehat{\mathfrak{A}}$$

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Improvements to $f : \mathbb{N} \to \mathbb{N}$

2005	$f(r) = 2^{2^{2^{\cdot^{\cdot^{\cdot 2}}}}}\}O(r)$	co-retracts $\widehat{\mathfrak{A}}_r \xrightarrow{\supseteq} \mathfrak{A}$
2017	$f(r) = 2^{O(r)}$	[R.'05] + AC^0 formula lower bounds
2018	$f(r) = \widetilde{O}(r^5)$	
2019	$f(r) = \widetilde{O}(r^3)$	
2025	$f(r) = r$	[R.'05] + Cai-Fürer-Immerman construction

***Exponential-rank* HPT on Finite Structures** [R.'05] + [R.'14, R.'17]

On finite structures, every hom-preserved FO[Arb] sentence with quant. rank r is equivalent to an existential-positive sentence with quant. rank $2^{O(r)}$

- Combines the previous result with a tight lower bound on the AC^0 **formula size** (= FO[Arb] quantifier rank) of DISTANCE-k CONNECTIVITY.
- Uses the fact that the **longest path** in a graph gives an ***exponential approximation*** of **tree-depth**: $\log(\mathbf{lp}(G)) \leq \mathbf{td}(G) \leq \mathbf{lp}(G)$

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***Poly-rank* HPT on Finite Structures** [R.'05] + [Li-Razborov-R.'14, R'17, Kawarabayashi-R.'18, Kush-R.'20, Czerwinski-Nadara-Pilipczuk '19]

On finite structures, every hom-preserved FO[Arb] sentence with quant. rank r is equivalent to an existential-positive sentence with quant. rank $\widetilde{O}(r^3)$

- Relies on a ***polynomial excluded-minor approximation*** of **tree-depth** (in terms of **longest path**, tallest complete **binary tree minor**, and **tree-width**).

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- Proof combines the original **HPT on Finite Structures** [R.'05] with a generalized **Cai-Fürer-Immerman construction** [CFI '94].

Generalized Cai-Fürer-Immerman Construction

Let $\mathfrak{A}$ be a finite structure with universe V, and let $\alpha : V \to \mathbb{Z}_2$.

Then $\mathbf{CFI}_\alpha(\mathfrak{A})$ is a structure—in the same relational signature—whose elements are pairs (v, λ) such that $v \in V$ and $\lambda : \mathrm{Nbd}(v) \to \mathbb{Z}_2$ and

$$\textstyle\sum_{w \in \mathrm{Nbd}(v)} \lambda(w) = \alpha(v).$$

For each t-ary relation $R \subseteq V^t$ in $\mathfrak{A}$, the corresponding relation in $\mathbf{CFI}_\alpha(\mathfrak{A})$ consists of t-tuples $((v_1, \lambda_1), \ldots, (v_t, \lambda_t))$ such that $(v_1, \ldots, v_t) \in R$ and

$$\lambda_i(v_j) = \lambda_j(v_i) \text{ for all } i, j \in \{1, \ldots, t\}.$$

OBS: The projection $(v, \lambda) \mapsto v$ is a homomorphism $\mathbf{CFI}_\alpha(\mathfrak{A}) \to \mathfrak{A}$.

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Key Lemma

Suppose $\mathfrak{A}$ is a core (i.e. every homomorphism $\mathfrak{A} \to \mathfrak{A}$ is an isomorphism).

Then there exists a homomorphism $\mathfrak{A} \to \mathbf{CFI}_\alpha(\mathfrak{A})$ if, and only if, $\sum_{u \in U} \alpha(u) = 0$ for every connected component $U \subseteq V$ (in the Gaifman graph of $\mathfrak{A}$).

Generalized Cai-Fürer-Immerman Construction

Well-known property of CFI structures [CFI '94]

If $\mathfrak{A}$ has **tree-depth** $> r$ (respectively, **tree-width** $> k$), then for all $\alpha, \beta : V \to \mathbb{Z}_2$, structures $\mathbf{CFI}_\alpha(\mathfrak{A})$ and $\mathbf{CFI}_\beta(\mathfrak{A})$ are indistinguishable by first-order sentences with **quantifier rank** r (respectively, **variable width** k).

Equi-rank HPT on Finite Structures [R.'25]

Suppose Φ is first-order sentence of quantifier rank r that is preserved under homomorphisms on finite structures. Then Φ is equivalent on finite structures to an $\exists^+$ sentence of quantifier rank r.

PROOF

- By the **HPT on Finite Structures**, the class $\mathrm{Mod}_{\mathrm{fin}}(\Phi)$ is generated by finitely many $\rightarrow$-incomparable ***minimal cores*** $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$.

  That is, for all finite structures $\mathfrak{A}$,

$$\mathfrak{A} \models \Phi \iff \bigvee_{i=1}^{m} (\mathfrak{C}_i \rightarrow \mathfrak{A}).$$

- By lemmas in [R.'05], it suffices to show that each $\mathfrak{C} \in \{\mathfrak{C}_1, \ldots, \mathfrak{C}_m\}$ has **tree-depth** $\leq r$.

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PROOF (continued)

- CLAIM: Every ***minimal core*** $\mathfrak{C} \in \mathrm{Mod}_{\mathsf{fin}}(\Phi)$ has **tree-depth** $\leq r$.
- Let V be the universe of $\mathfrak{C}$ and let $\mathsf{even}, \mathsf{odd} : V \to \mathbb{Z}_2$ such that

$$\mathsf{even} \equiv 0 \quad \text{and} \quad \textstyle\sum_{v \in V} \mathsf{odd}(v) = 1.$$

- By the **Key Lemma**, $\mathfrak{C} \to \mathbf{CFI}_{\mathsf{even}}(\mathfrak{C})$ and $\mathfrak{C} \not\to \mathbf{CFI}_{\mathsf{odd}}(\mathfrak{C}) \to \mathfrak{C}$.
- Therefore, $\mathbf{CFI}_{\mathsf{even}}(\mathfrak{C}) \models \Phi$ (since Φ is hom-preserved) and $\mathbf{CFI}_{\mathsf{odd}}(\mathfrak{C}) \not\models \Phi$ (since $\mathfrak{C}$ is a minimal core).
- Since $\mathbf{CFI}_{\mathsf{even}}(\mathfrak{C})$ and $\mathbf{CFI}_{\mathsf{odd}}(\mathfrak{C})$ are distinguished by a first-order sentence of quantifier rank r, it follows that $\mathfrak{C}$ has **tree-depth** $\leq r$ (by the well-known property of CFI structures).

Q.E.D.

A refinement of this argument establishes a stronger version of our equi-rank result in terms of multiple syntactic parameters:

***Equi-Resource* HPT on Finite Structures** [R.25]

On finite structures, every homomorphism-preserved sentence is equivalent to an existential-positive sentence with the same ***quantifier rank***, ***number of variables***, and ***alternation type***.