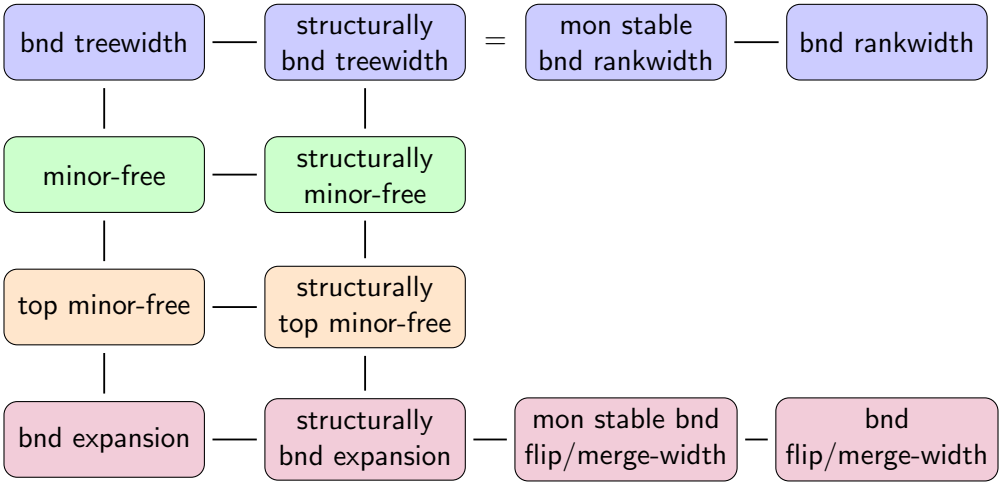
Low rank MSO

Wojciech Przybyszewski

Joint work with Mikołaj Bojańczyk, Michał Pilipczuk, Marek Sokołowski, and Giannos Stamoulis

Finite and Algorithmic Model Theory 2025

Map of the universe (a fragment)



Topological minor free

Definition.

A graph G contains clique K_t as a *topological minor* if there are t vertices $v_1, \ldots, v_t \in V(G)$ connected by a pairwise internally vertex-disjoint paths. A graph class $\mathscr{C}$ is *topological minor free* if there exists t such that every graph $G \in \mathscr{C}$ does not contain K_t as a topological minor.

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What is the dense analog of topological minor free graphs?

FO+CONN logic

FO+CONN logic [Bojańczyk, 2021] [Schirrmacher, Siebertz, Vigny, 2021] is an extension of FO by a simple mechanism for expressing connectivity: we allow the usage of predicates $\mathrm{conn}_k(s, t, a_1, \ldots, a_k)$, for every $k \in \mathbb{N}$, that verify the existence of a path connecting vertices s and t that avoids vertices $a_1, \ldots, a_k$.

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Theorem. [Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, Vigny, 2021]

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FO+CONN logic in a sense allows quantification over small separators – instead of saying there exist k vertices that separate s and t we can say there exists a partition $X \cup Y = V(G)$ with $|X \cap Y| \leq k$, $s \in X$, and $t \in Y$, such that there are no edges between $X - Y$ and $Y - X$.

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$$\mathrm{rk}(X) = |\{N_G(v) \cap \overline{X} \mid v \in X\}|$$

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More formally, instead of the quantifiers $\exists X$ and $\forall X$ we use $\exists_{\mathrm{rk} \leq k} X$ and $\forall_{\mathrm{rk} \leq k} X$, where X is a set of vertices and $k \in \mathbb{N}$.

Low rank MSO and sparse graphs

Theorem. [Bojańczyk, Pilipczuk, P., Sokołowski, Stamoulis]

Let $\mathscr{C}$ be a weakly sparse ($= K_{t,t}$-subgraph-free for some t) class of graphs. Then low rank MSO and separator logic have the same expressive power on $\mathscr{C}$.

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Lemma.

Let G be a graph that does not contain the complete bipartite graph $K_{t,t}$ as a subgraph. Let X be a set of rank at most r in G. Then there is a set $S \subseteq V(G)$ of size at most $\mathrm{const}(r,t)$ such that X is a union of connected components of $G - S$ and elements of S.

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Nota bene: the theorem gives a model-checking algorithm for low rank MSO on weakly sparse classes of graphs, that is polynomial for a fixed formula.

Flip-connectivity logic

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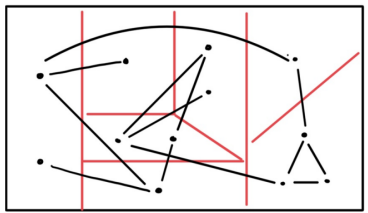

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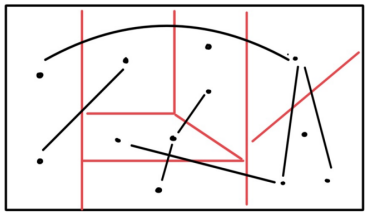

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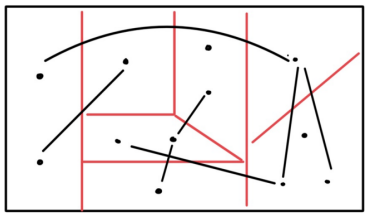

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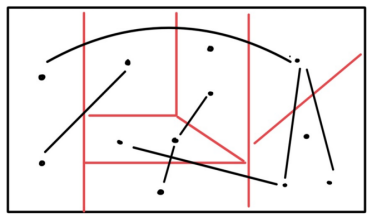

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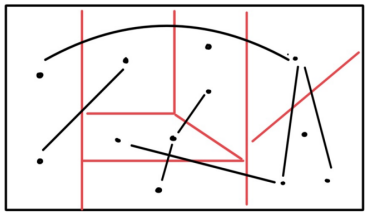

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Is low rank MSO equivalent to FO+FLIPCONN logic on the class of all graphs? NO!
[Figure at the blackboard]

Low rank MSO on classes of bounded VC dimension

Theorem. [Bojańczyk, Pilipczuk, P., Sokołowski, Stamoulis]

Let $\mathscr{C}$ be a class of graphs of bounded VC dimension. Then low rank MSO and flip-connectivity logic have the same expressive power on $\mathscr{C}$.

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Let $\mathscr{C}$ be a class of graphs of bounded VC dimension. Then low rank MSO and flip-connectivity logic have the same expressive power on $\mathscr{C}$.

Again we need a lemma that describes sets of low rank in graphs of bounded VC dimension – for this we use a result of [Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, Toruńczyk, 2022].

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Theorem. [Bojańczyk, Pilipczuk, P., Sokołowski, Stamoulis]

Low rank MSO and flip-reachability logic have the same expressive power on the class of all graphs.

Open questions

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Thank you!