# Symmetric Algebraic Circuits and Homomorphism Polynomials

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### Determinant

#### Permanent

$$\det_n = \sum_{\pi \in \mathbf{Sym}_n} \operatorname{sgn}(\pi) \cdot x_{1\pi(1)} \cdots x_{n\pi(n)}$$

Circuit complexity:  $\mathcal{O}(n^4)$ .

 $\mathsf{perm}_n = \sum_{\pi \in \mathbf{Sym}_n} x_{1\pi(1)} \cdots x_{n\pi(n)}$ 

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### "VP = VNP?" is the question "Does $(perm_n)_{n \in \mathbb{N}}$ admit polynomial-size algebraic circuits?"

# Theorem (Dawar, Wilsenach; 2020)

- 1. There are polynomial-size symmetric circuits for  $(\det_n)_{n \in \mathbb{N}}$ .
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In this work: Complete characterisation of polynomials with polynomial-size symmetric circuits.

- Let  $\mathcal{X}_{n,m} \coloneqq \{x_{ij} \mid i \in [n], j \in [m]\}.$
- $\operatorname{Sym}_n \times \operatorname{Sym}_m$  acts on  $\mathcal{X}_{n,m}$ : For  $(\pi, \sigma) \in \operatorname{Sym}_n \times \operatorname{Sym}_m$ , it is  $(\pi, \sigma)(x_{ij}) = x_{\pi(i)\sigma(j)}$ .
- An algebraic circuit C over  $\mathcal{X}_{n,m}$  is  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric if the action on  $\mathcal{X}_{n,m}$  extends to automorphisms of C.

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- Every  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric polynomial  $p \in \mathbb{Q}[\mathcal{X}_{n,m}]$  defines a  $\mathbb{Q}$ -valued function on *bipartite* (n, m)-vertex graphs.
- **Example:**  $perm_n(G)$  is the number of perfect matchings in a bipartite (n, n)-vertex graph G.

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Let *F* be a bipartite graph. For each  $n, m \in \mathbb{N}$ , the following polynomial evaluated in an (n, m)-vertex graph *G* counts the number of homomorphisms from *F* to *G*.

$$\hom_{F,n,m} := \sum_{h: V(F) \to [n] \uplus [m]} \prod_{ab \in E(F)} X_{h(a)h(b)}.$$

Let  $\mathfrak{T}_{n,m}^k$  be the set of  $\mathbb{Q}$ -linear combinations of polynomials hom<sub>*F<sub>i</sub>*,*n*,*m*</sub> where all *F<sub>i</sub>* have *treewidth* at most *k*.

#### Theorem

For every family of polynomials  $p_{n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$ , the following are equivalent:

- 1. there exists a constant  $k \in \mathbb{N}$  such that  $p_{n,m} \in \mathfrak{T}_{n,m}^k$  for all  $n, m \in \mathbb{N}$ ,
- 2. the  $p_{n,m}$  admit  $Sym_n \times Sym_m$ -symmetric circuits of size polynomial in n + m.

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- in FPT, if it is expressible as a linear combination of homomorphism counts from bounded-treewidth graphs,
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### Differences to our result:

- Syntactic complexity of polynomials vs computational complexity of the counting functions.
- Our pattern graphs have no fixed size.
- Our lower bound is unconditional.

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- Conjecture: In general, min{vc(F), vc(F)} is the criterion for tractability of subgraph polynomials.

If  $p_{n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$  admits poly-size symmetric circuits, then the counting width (Weisfeiler-Leman dimension) of the function defined by  $p_{n,m}$  on bipartite graphs is bounded.

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### Open question: Is the converse true?

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- "Symmetric VP" can be characterised as the class of all polynomials expressible via *bounded-treewidth homomorphism counts*.
- In special cases, such as subgraph polynomials, this translates to explicit criteria for super-polynomial lower bounds.
- **Application:** A (conditional) complexity dichotomy for the *immanant* polynomials due to Curticapean (2021) holds unconditionally for symmetric circuits.