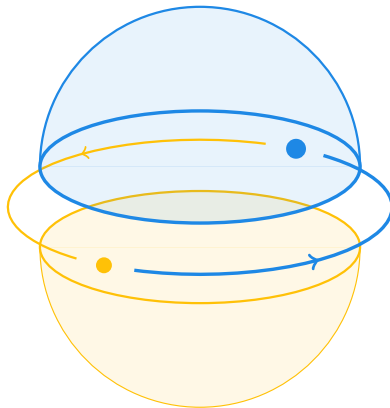


A topological proof of the H -colouring dichotomy

Jakub Opršal et al.



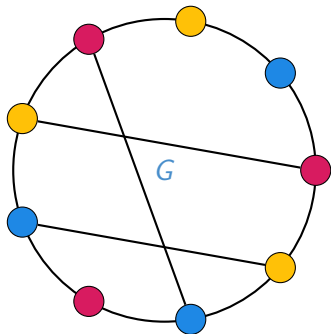
joint work with Sebastian Meyer (TU Dresden)



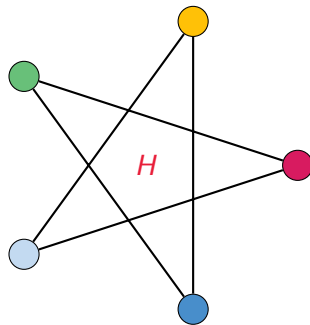
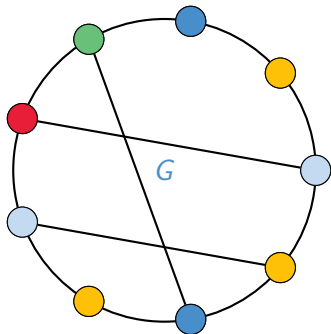
UNIVERSITY OF
BIRMINGHAM

Part I. Why?

Graph colouring



Graph colouring



Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a **graph homomorphism** $G \rightarrow H$ is a mapping $h: V_G \rightarrow V_H$ that preserves edges,

$$uv \in E_G \Rightarrow h(u)h(v) \in E_H.$$

Example. A **colouring** of a graph G with k colours is just a homomorphism $c: G \rightarrow K_k$.

Brakensiek–Guruswami conjecture and the H -colouring dichotomy

Conjecture [Brakensiek, Guruswami, 2018].

For every $k > 0$ and $c > 2$, it is NP-complete to distinguish graphs that map homomorphically to $C_{(2k+1)}$ from those that are not c -colourable.

- [1] Krokhn, O (2019). The complexity of 3-colouring H -colourable graphs. *FOCS 2019*.
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Theorem [Avvakumov et al., 2025].

Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ with 4 colours is NP-complete.

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H -colouring

Fix a graph H (called *template*). Given a graph G , decide whether there is a homomorphism $G \rightarrow H$.

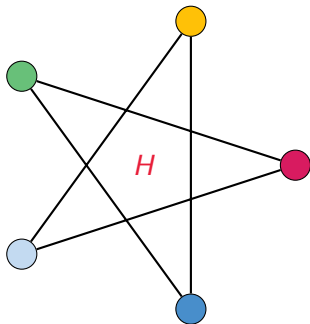
Theorem [Hell & Nešetřil, 1990].

Unless $P = NP$, the only (non-trivial) graph H -colouring problem that is solvable in polynomial time is 2-colouring.

Outline of the proof

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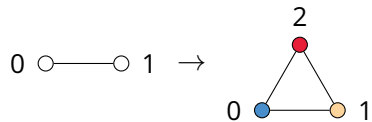


1. Identify which problems are **NP-hard** using the algebraic approach to the *constraint satisfaction problem*.
2. If H -colouring is **not NP-hard**, show that its **solution spaces** are *component-wise contractible*.
3. Use Brouwer's **fixed-point theorem** to show that H has a loop if H is not bipartite.

Part II. What the ... is the solution space of H -colouring?

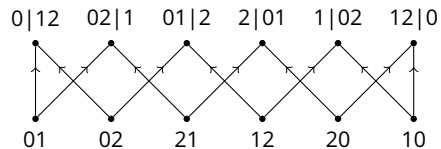
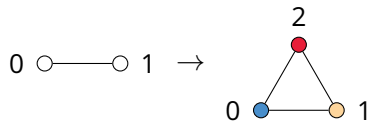
Solution spaces

$$K_2 \rightarrow K_3$$



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A **multihomomorphism** is a function

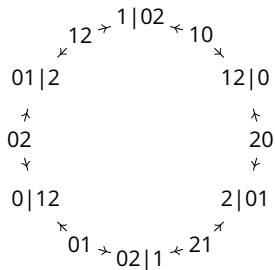
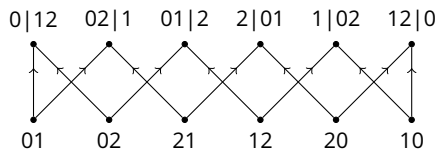
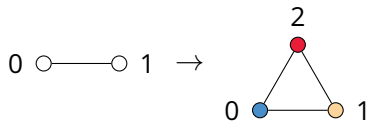
$$f: V(G) \rightarrow 2^{V(H)} \setminus \{\emptyset\}$$

such that, for all edges $uv \in E(G)$, we have that

$$f(u) \times f(v) \subseteq E(H).$$

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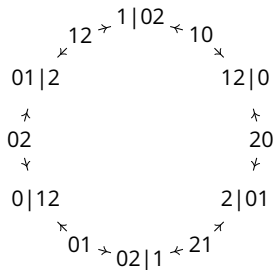
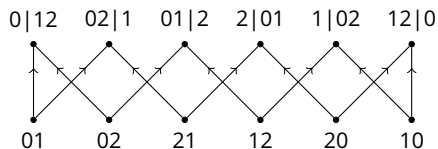
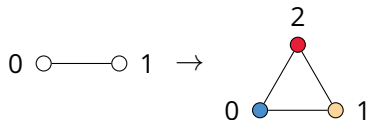
The space $\text{Hom}(G, H)$

Example. $\text{Hom}(K_2, K_3) \simeq S^1$.

$\text{Hom}(K_2, H)$ is homotopy-equivalent to the space used by [Lovász, 1977] to provide lower bounds on chromatic number of *Kneser graphs*.

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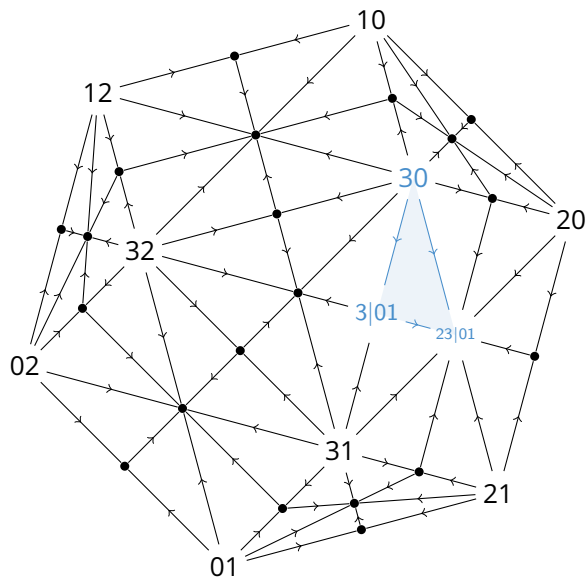
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$$01 \leq 0|12 \leq 0|123$$

which creates **2-dimensional** faces in $\text{Hom}(K_2, K_4)$.

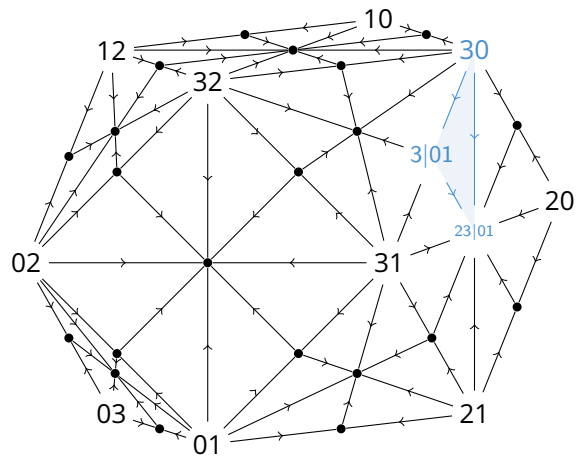
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4-colourings of K_2



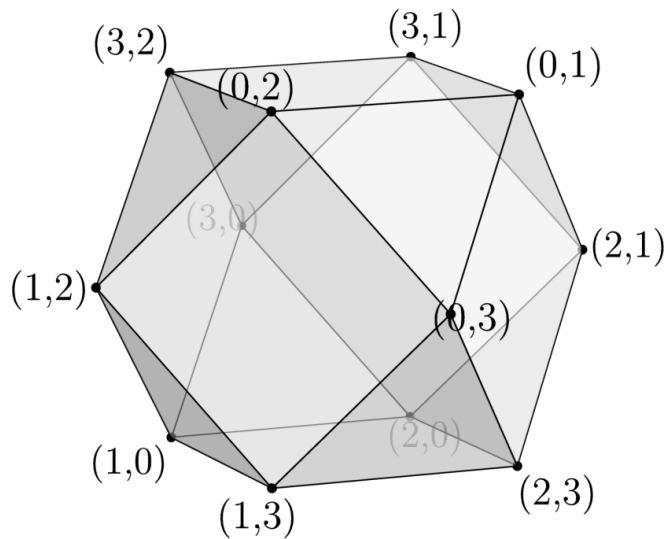
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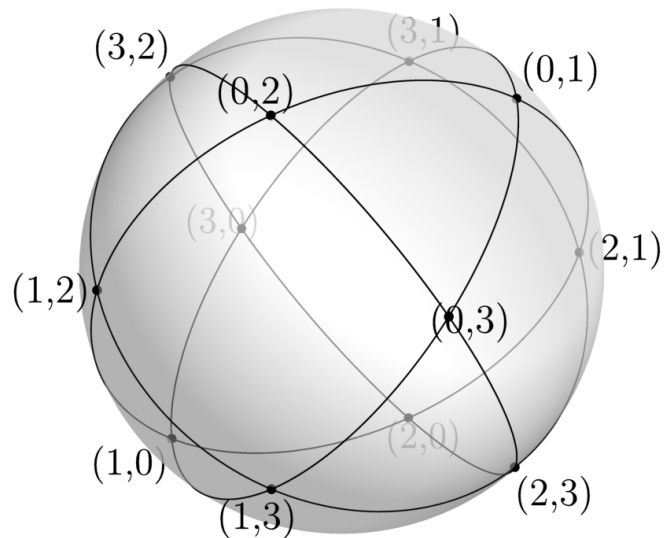
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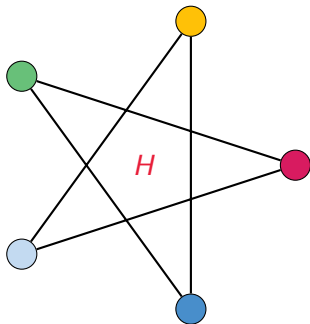
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Part III. A proof

Outline of the proof

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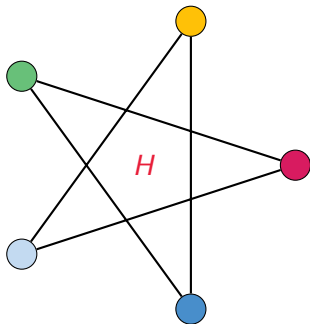


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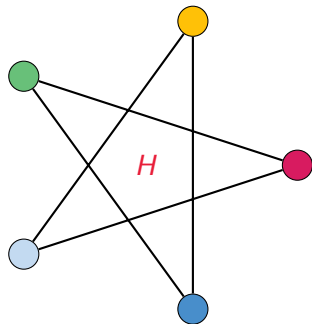


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Algebraic approach to the constraint satisfaction problem

An operation $t: A^n \rightarrow A$ is Taylor

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for all $x, y \in A$.

$t: A^n \rightarrow A$ is idempotent if $t(x, \dots, x) = x$.

Theorem (CSP Dichotomy).

A CSP with a finite template A is either

1. admits a Taylor homomorphism $t: A^n \rightarrow A$, and is in P [Bulatov, 2017; Zhuk, 2017]; or
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A topological space X is called contractible if it is homotopy equivalent to a point $\{*\}$. For us, this is equivalent to $\pi_n(X) = 0$ for all $n \geq 0$.

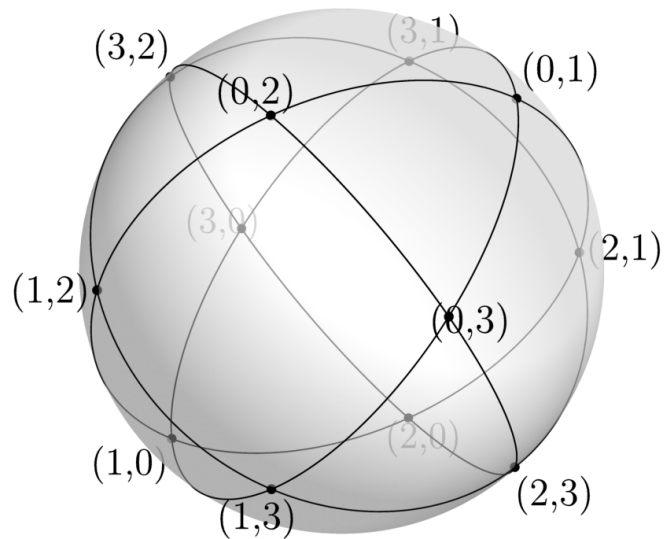
Theorem [Larose, Zádori, 2005].

Every connected finite poset that admits a monotone Taylor operation is contractible.

Theorem [Meyer, 2024; Meyer, O, 2025].

If H admits a Taylor homomorphism, then $\text{Hom}(G, H)$ is component-wise contractible for every G .

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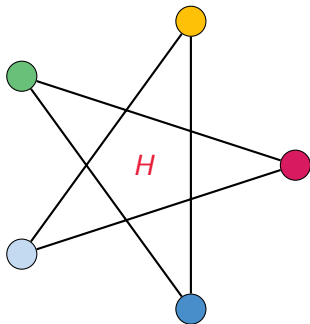


Hence, 4-colouring is NP-hard!

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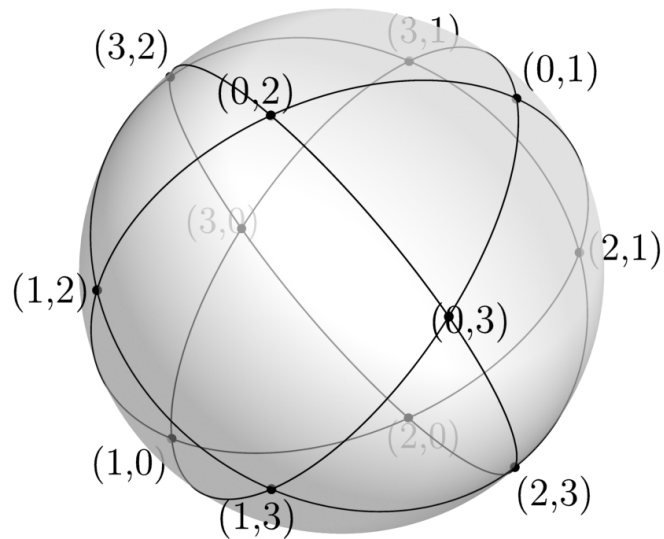
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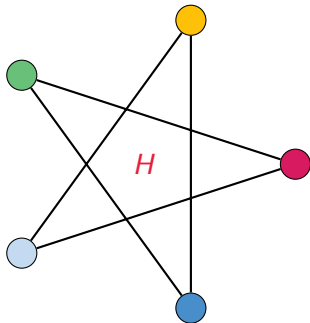


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Proof. Assume that H is not-bipartite, and consider the space $X = \text{Hom}(K_2, H)$.

Observe that the space admits a fixed-point free \mathbb{Z}_2 -action $\phi: X \rightarrow X$ that for each multihomomorphism m flips the values of $m(0)$ and $m(1)$.

If H is not-bipartite then ϕ fixes a connected component of X . Indeed, if uv is an edge of an odd cycle of H then uv is connected to $vu = \phi(uv)$.

If H admitted a Taylor homomorphism, $\text{mhom}(K_2, H)$ would admit a lax-Taylor operation, and all its connected component would be contractible.

Hence, ϕ which acts on the component of uv has a fixed point by the Brouwer's fixed point theorem, hence H has a loop. ■

How does the topology of the solution space influence the complexity of a computational problem?

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A *constraint satisfaction problem* is *NP-complete*, unless each connected component of the solution space is *contractible* (i.e., topologically trivial).

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