

Infinite-domain CSPs

$\mathbb{A} = (A; R_1, \dots, R_k)$ fixed *relational* structure

Definition (pp-model checking). The **primitive positive model checking** of \mathbb{A} is the following problem:

Input: an $\{\exists, \wedge\}$ -sentence $\varphi := \exists x_1 \dots \exists x_n \bigwedge_{i=1}^m \varphi_i$

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Boolean conjunctive query $Q()$

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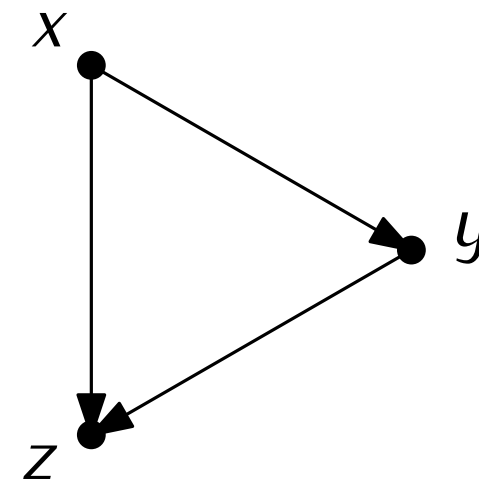
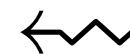
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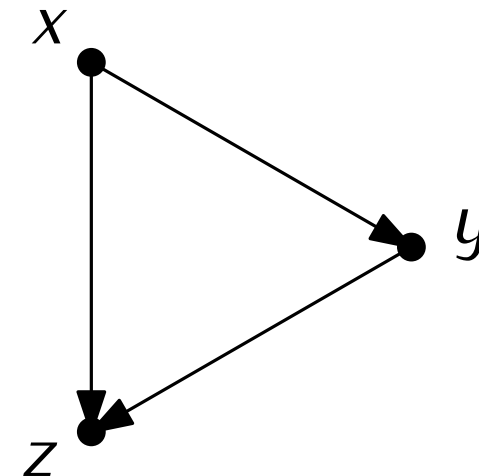
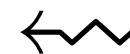
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- \mathbb{A} is called the **template** of the problem
- $\text{Hom}(\mathbb{A})$ also called the **constraint satisfaction problem** (CSP) of \mathbb{A}

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Definition (Label cover). A label cover instance is a directed graph where every vertex v is labelled by a set S_v and every edge (v, w) is labelled by a function $S_v \rightarrow S_w$.

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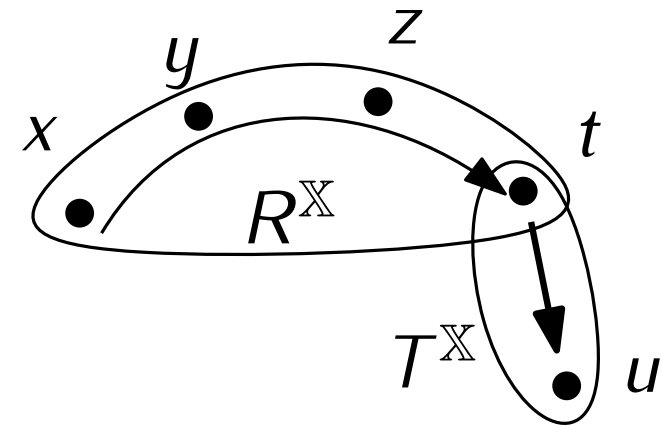
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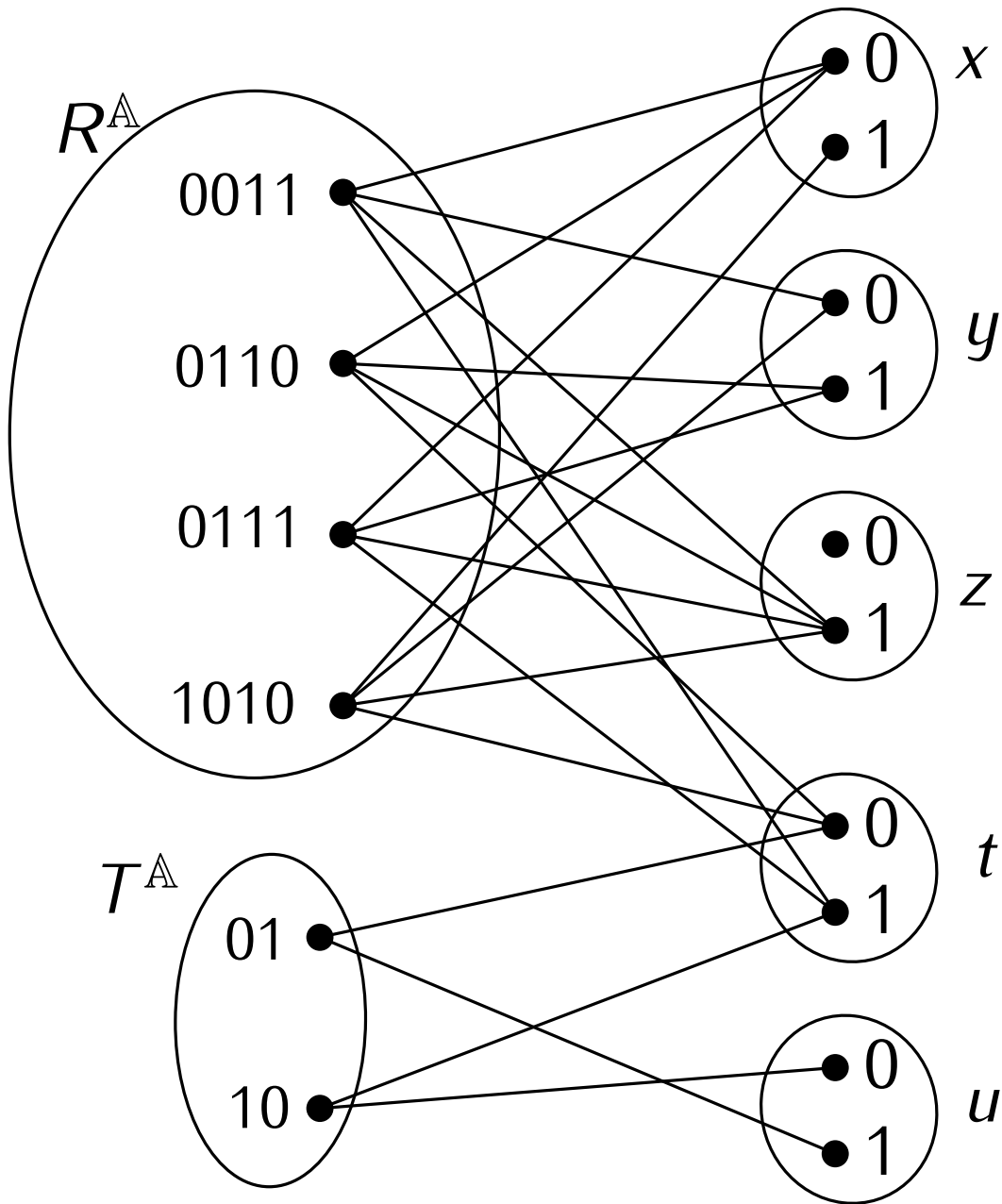
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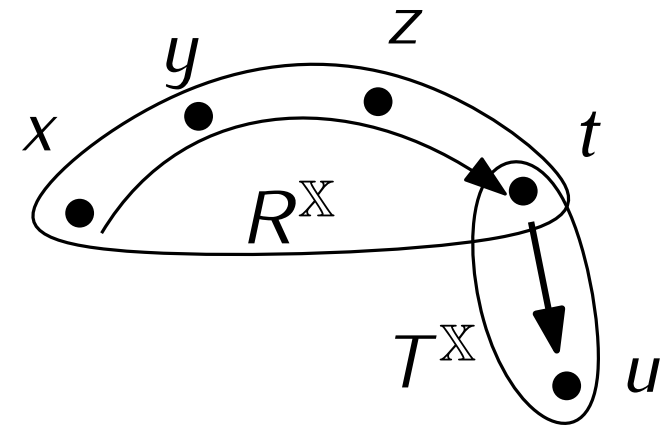
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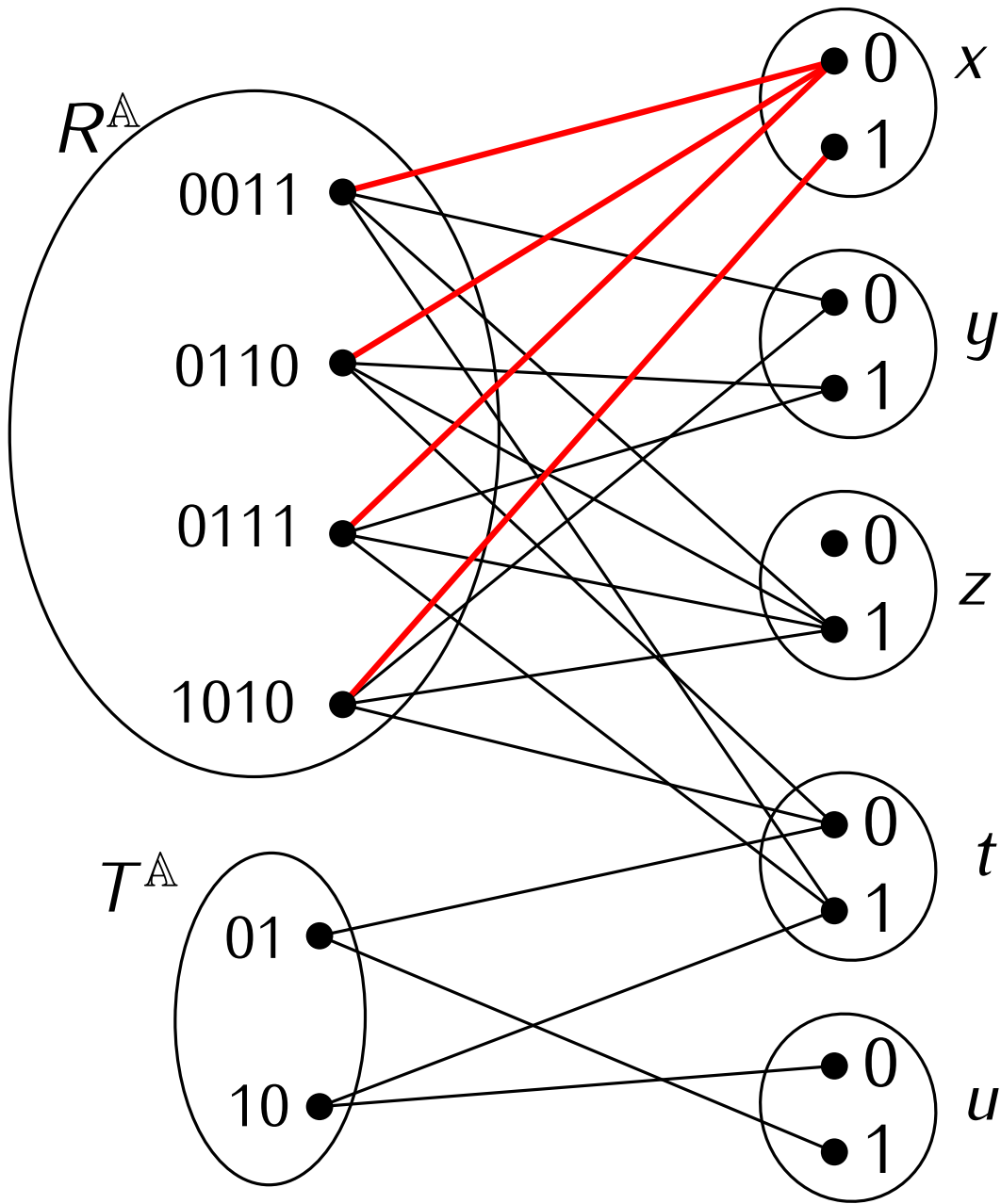
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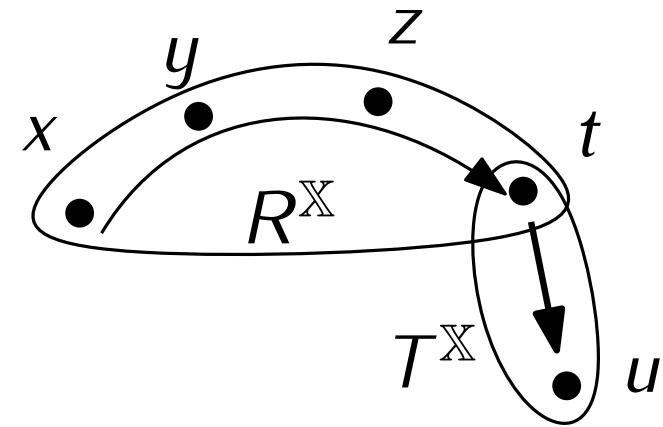
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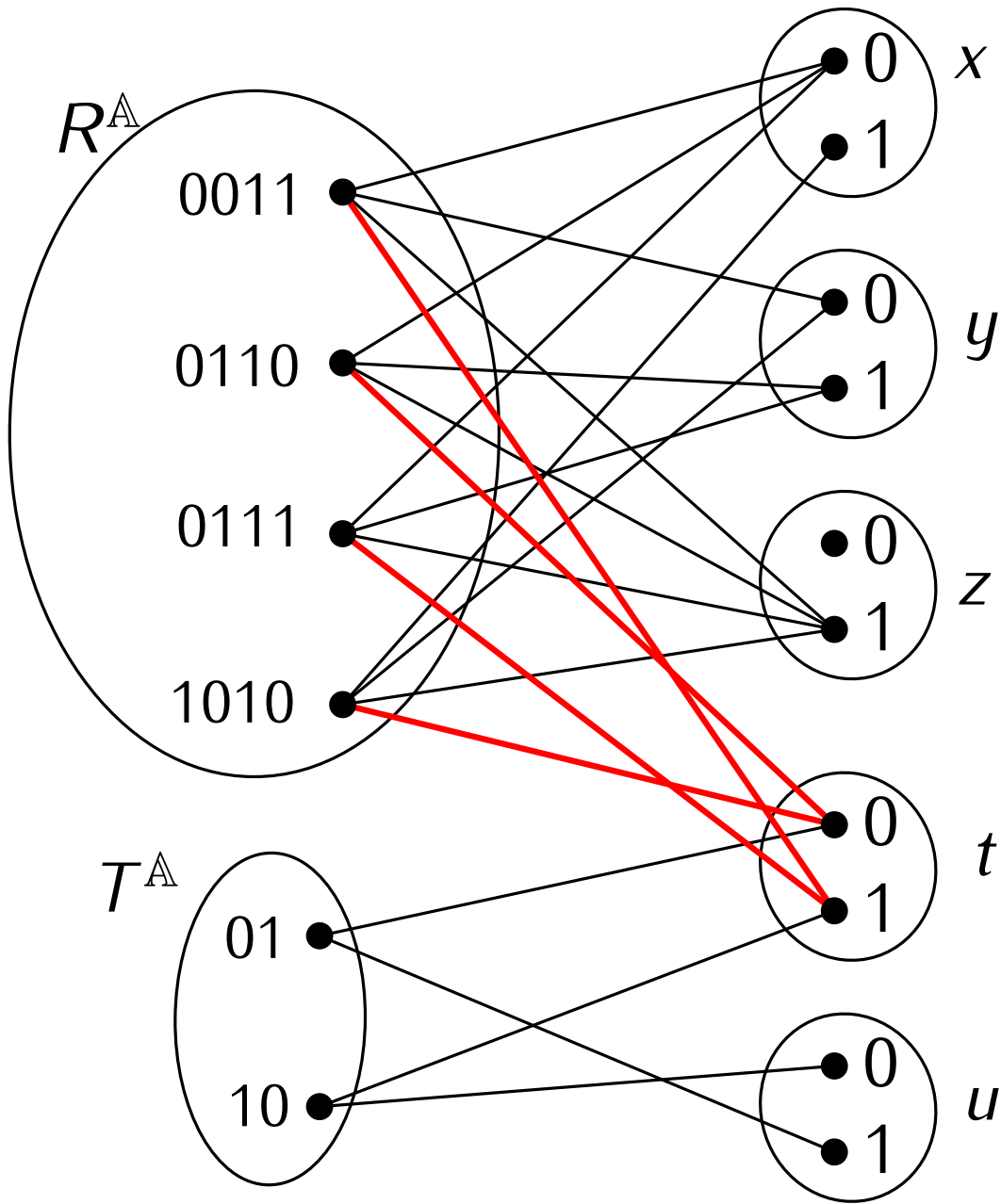
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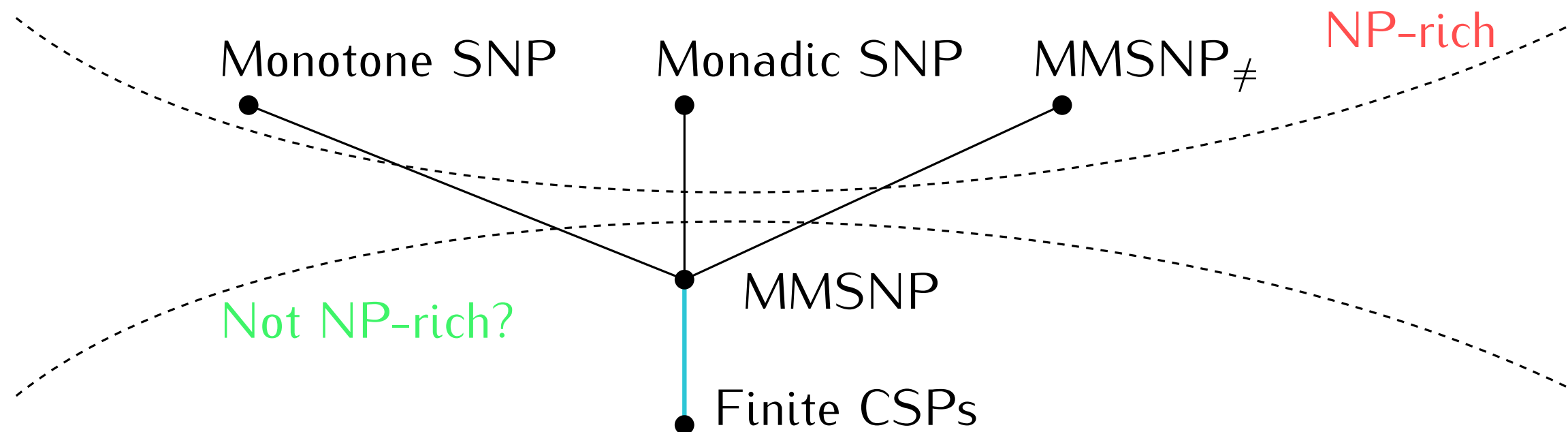
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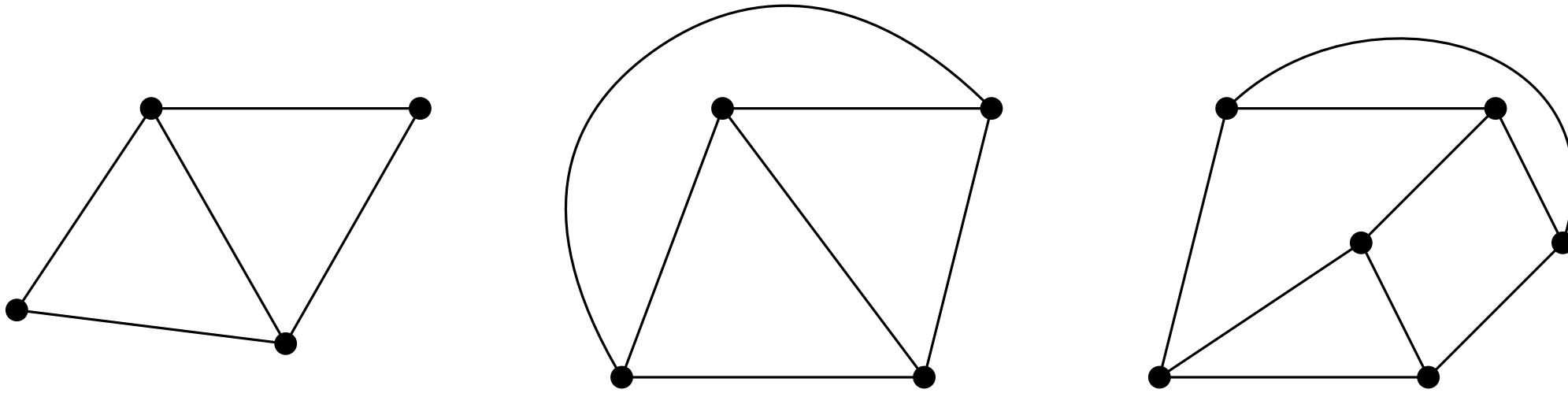
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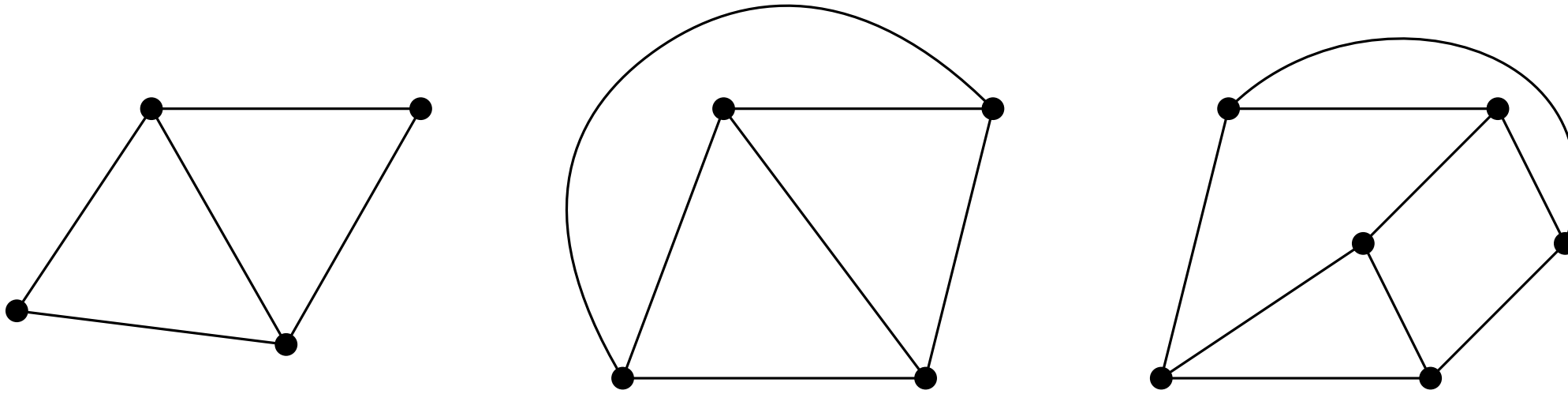
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- Take $\mathbb{A} = \bigsqcup_{\mathbb{X} \in \mathcal{C}} \mathbb{X}$
- Can be a useful point of view, but most likely not
- Call \mathcal{C} a **CSP** if it satisfies those properties

Goal: understand finite 3-colorable graphs

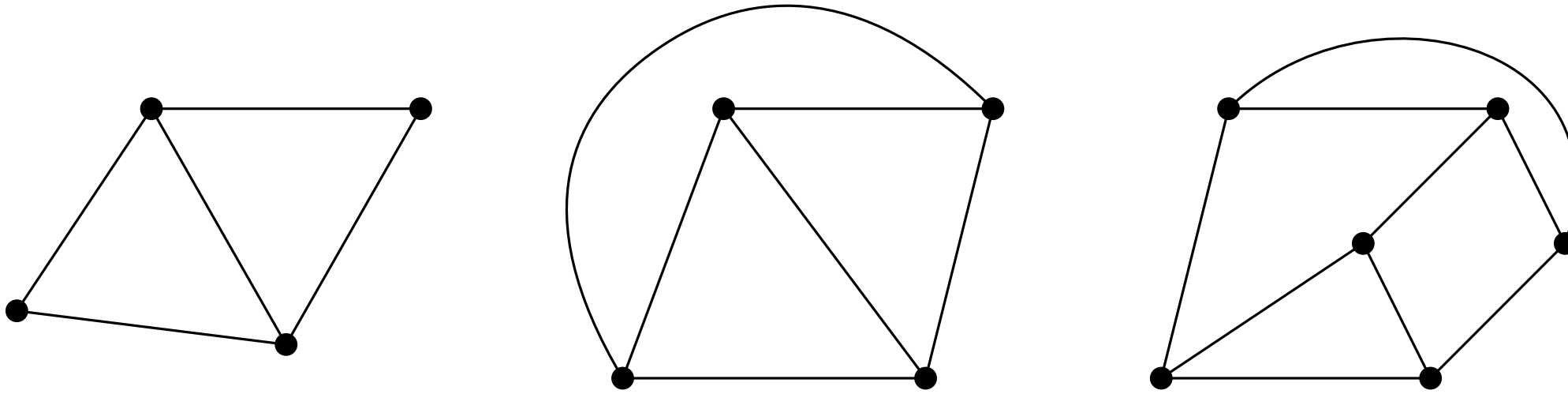


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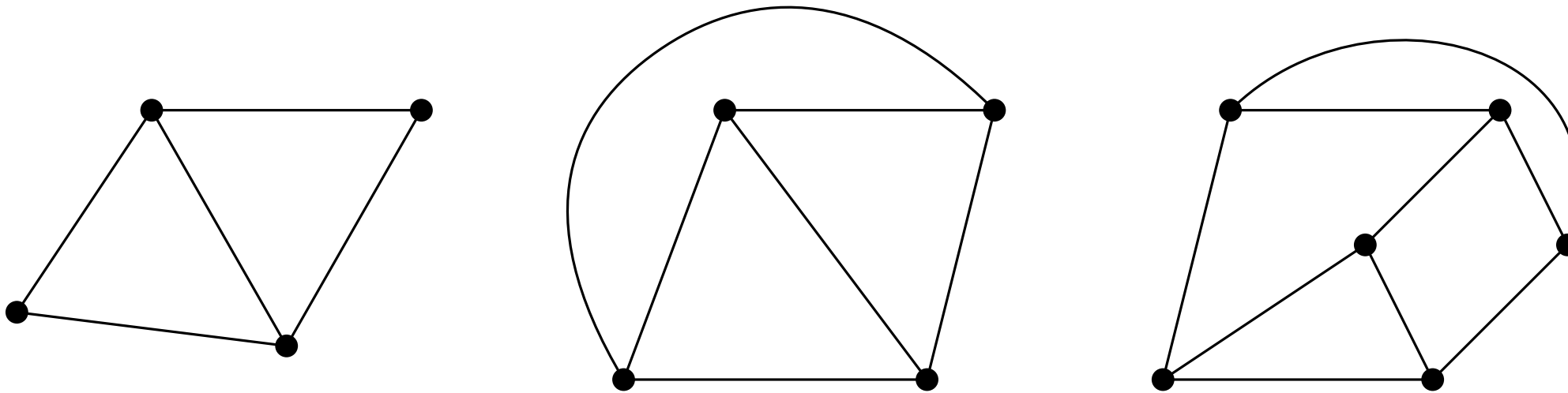
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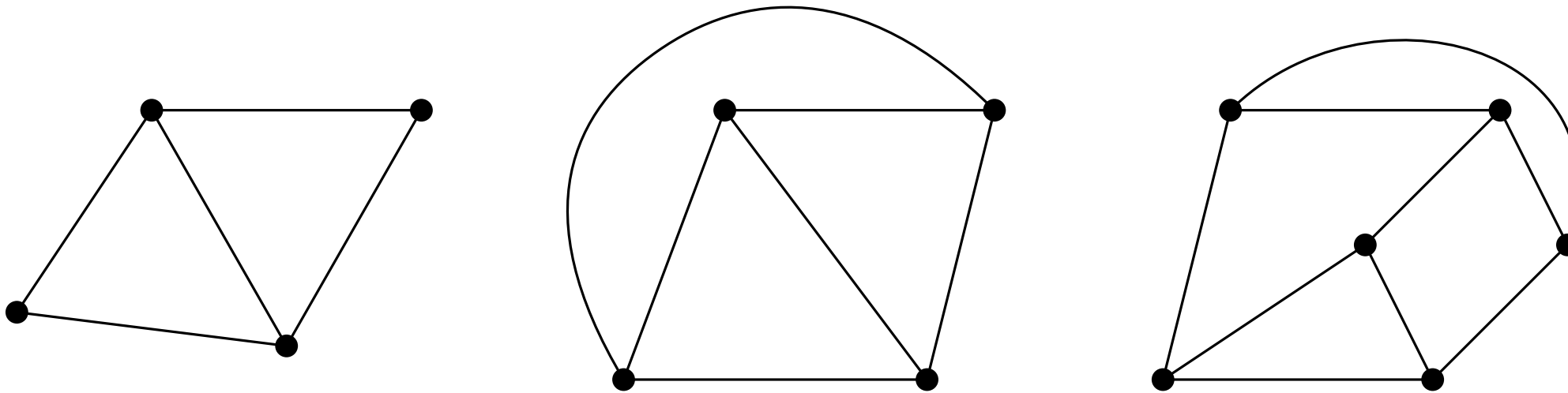


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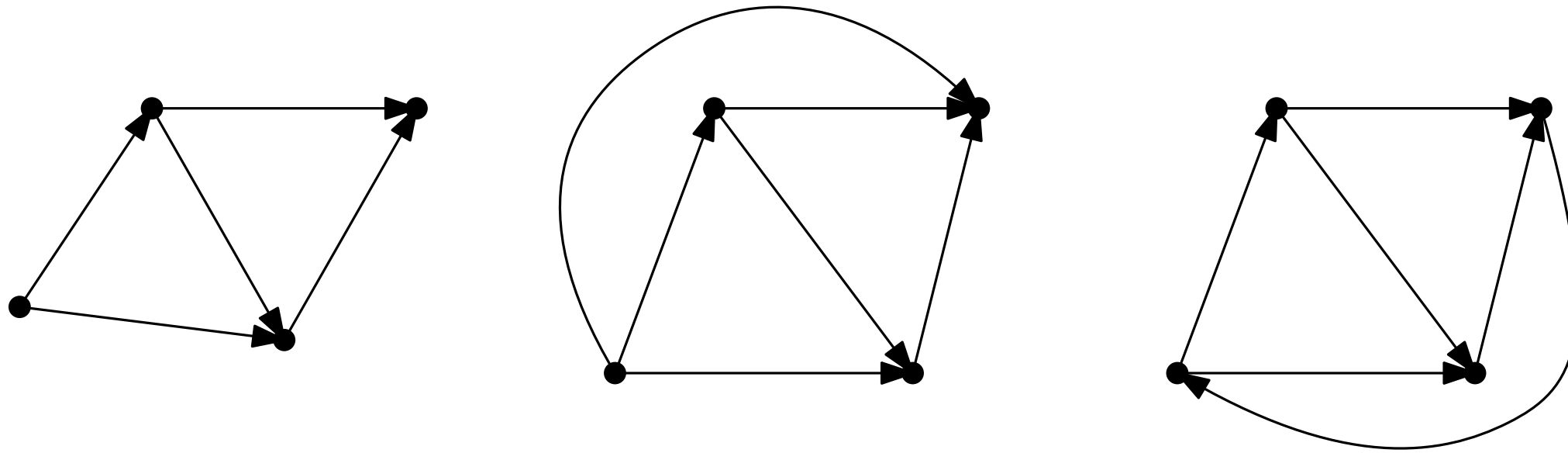
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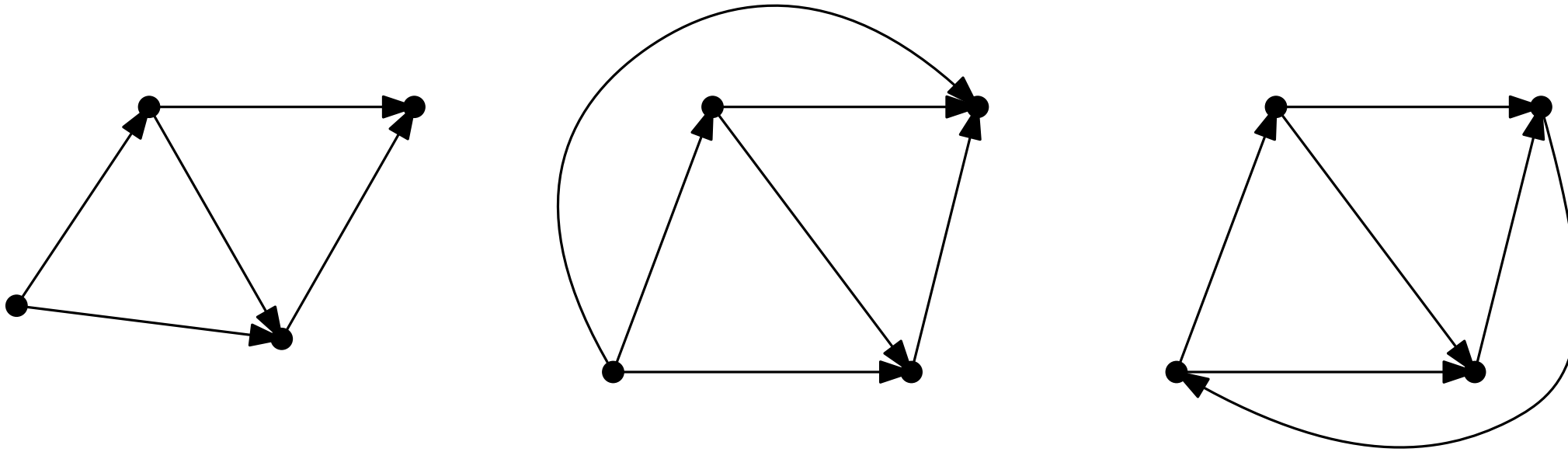


Step 4: $G^* \rightleftharpoons K_3$ so $\text{CSP}(G^*) = \text{CSP}(K_3)$

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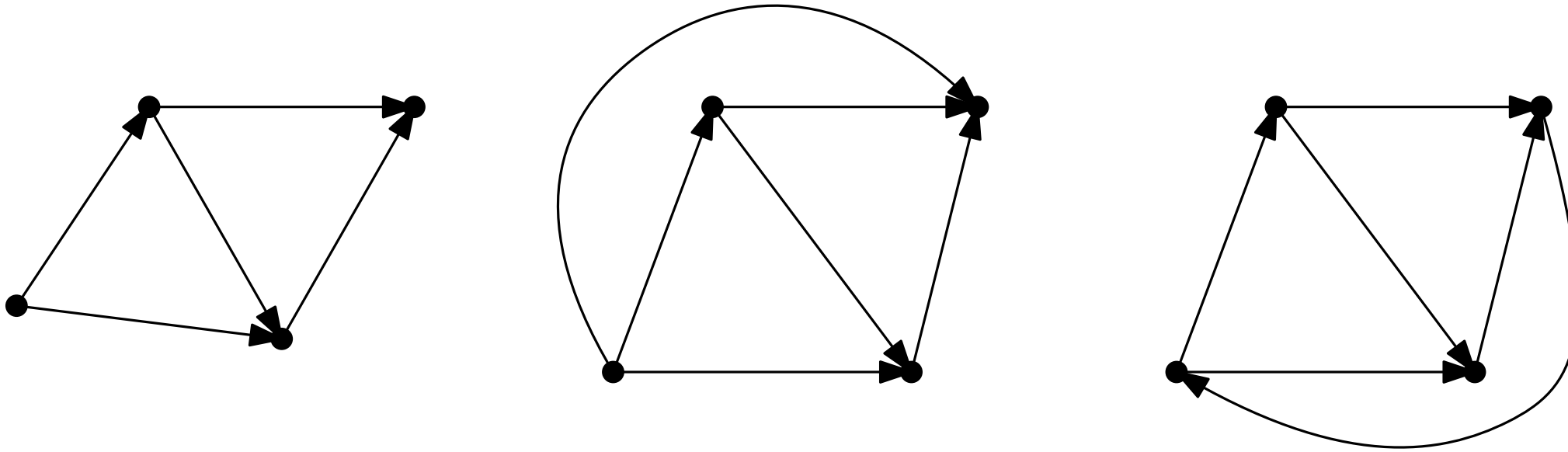


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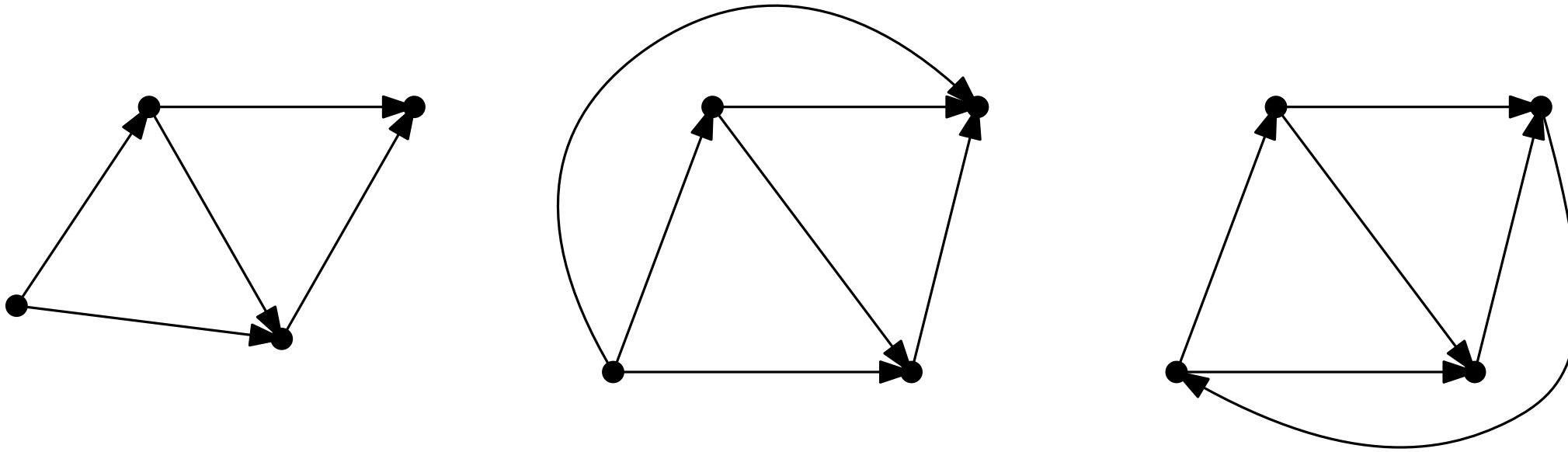
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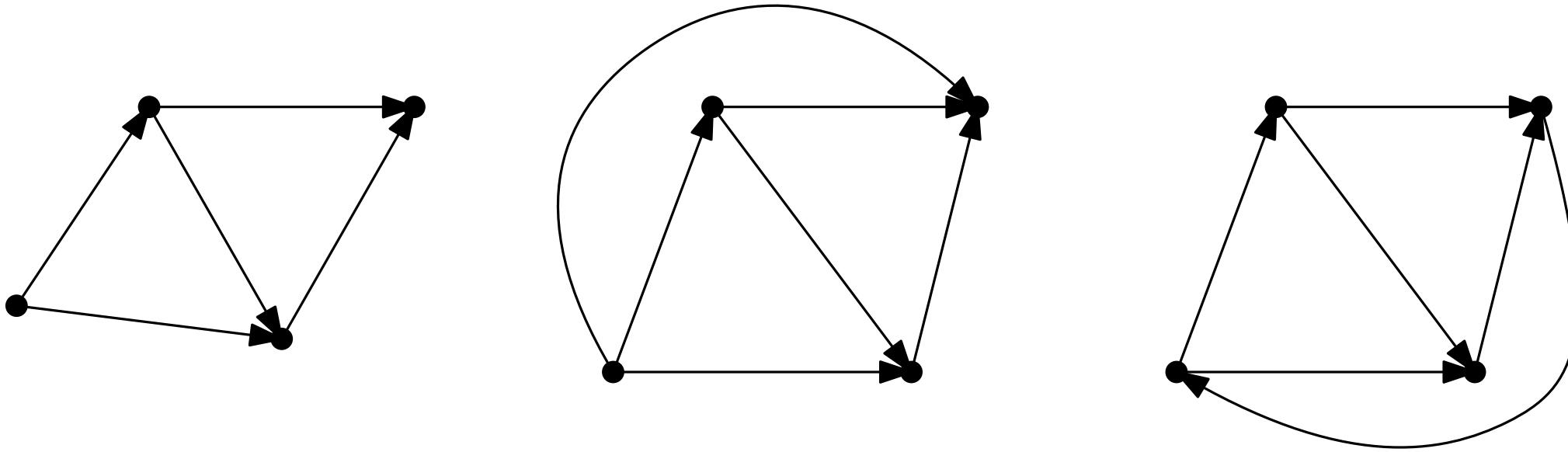


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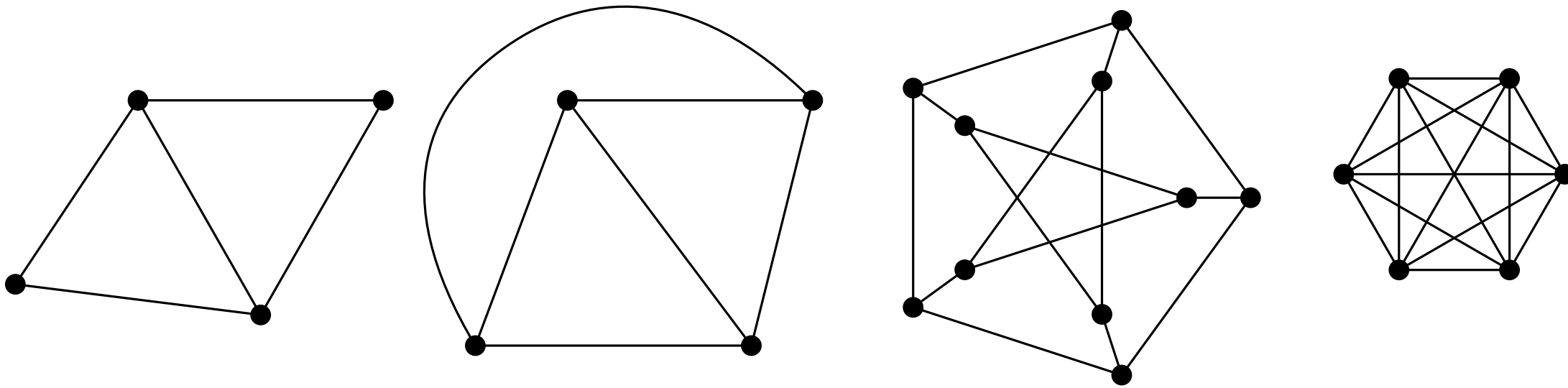
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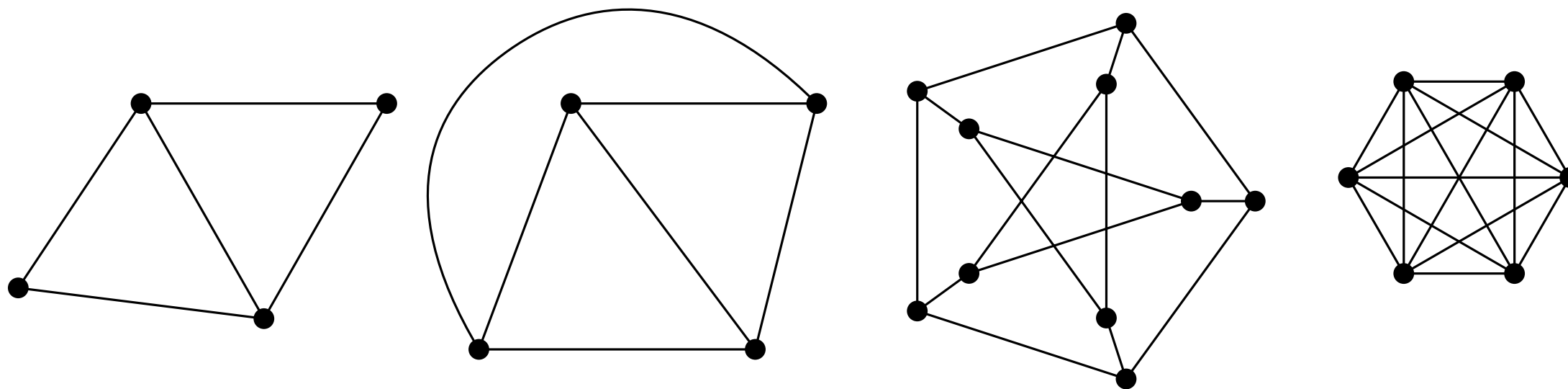
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Step 4: $\text{CSP}(G^*) = \text{CSP}(\mathbb{N}; <)$. Acyclic digraphs are those with a topological ordering.



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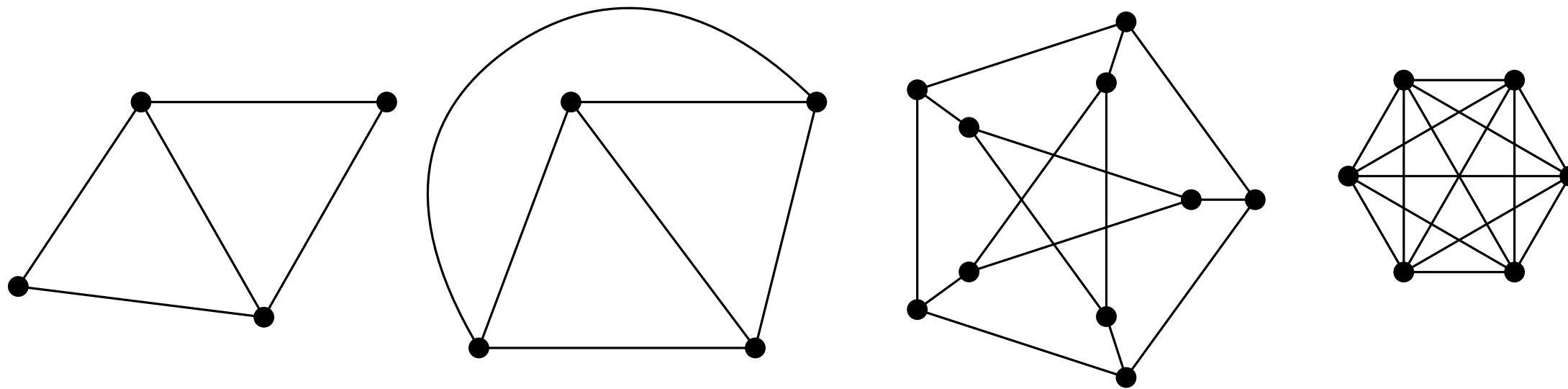
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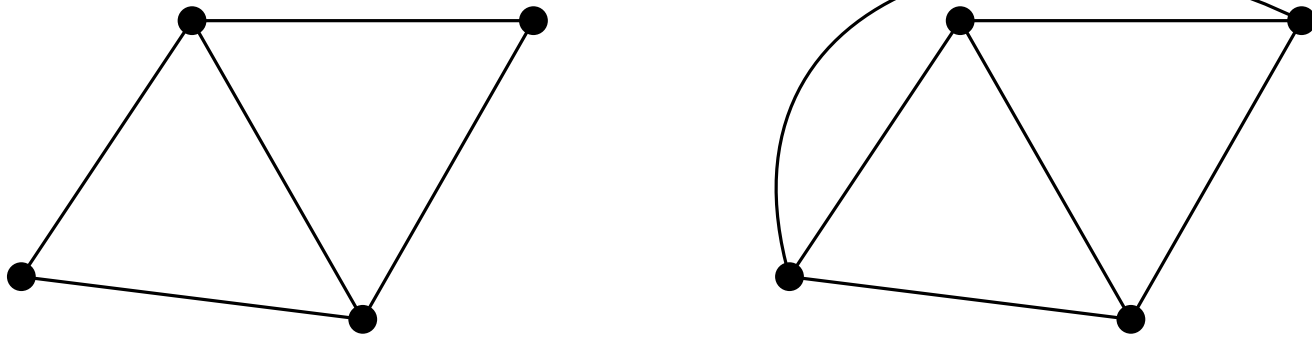
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Goal: understand graphs that can be oriented in a way that every triangle becomes a directed cycle



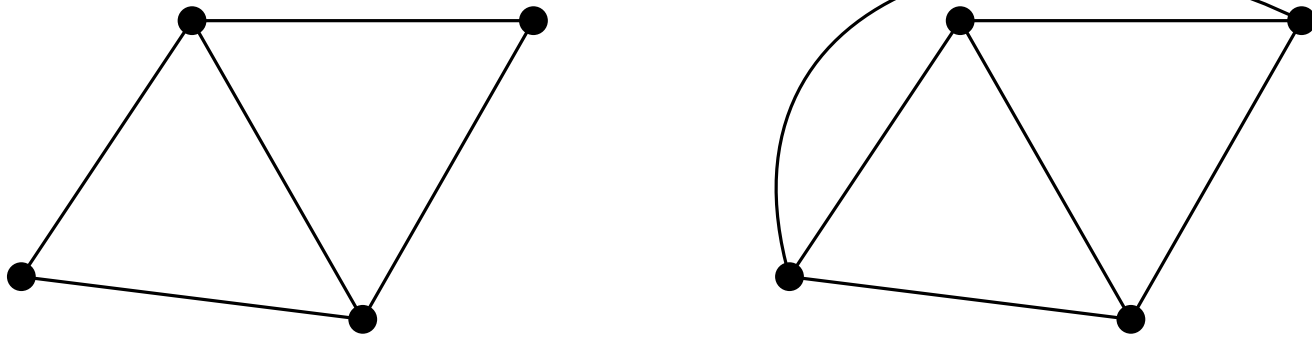
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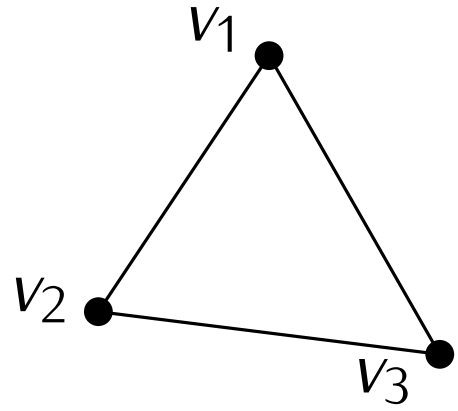
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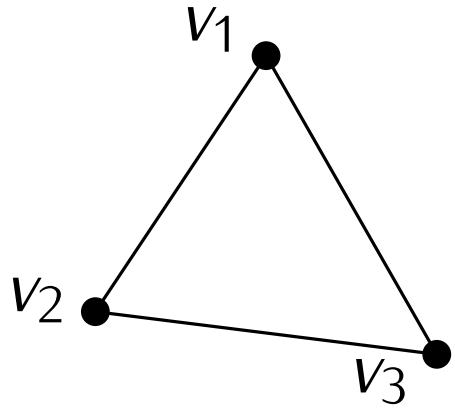


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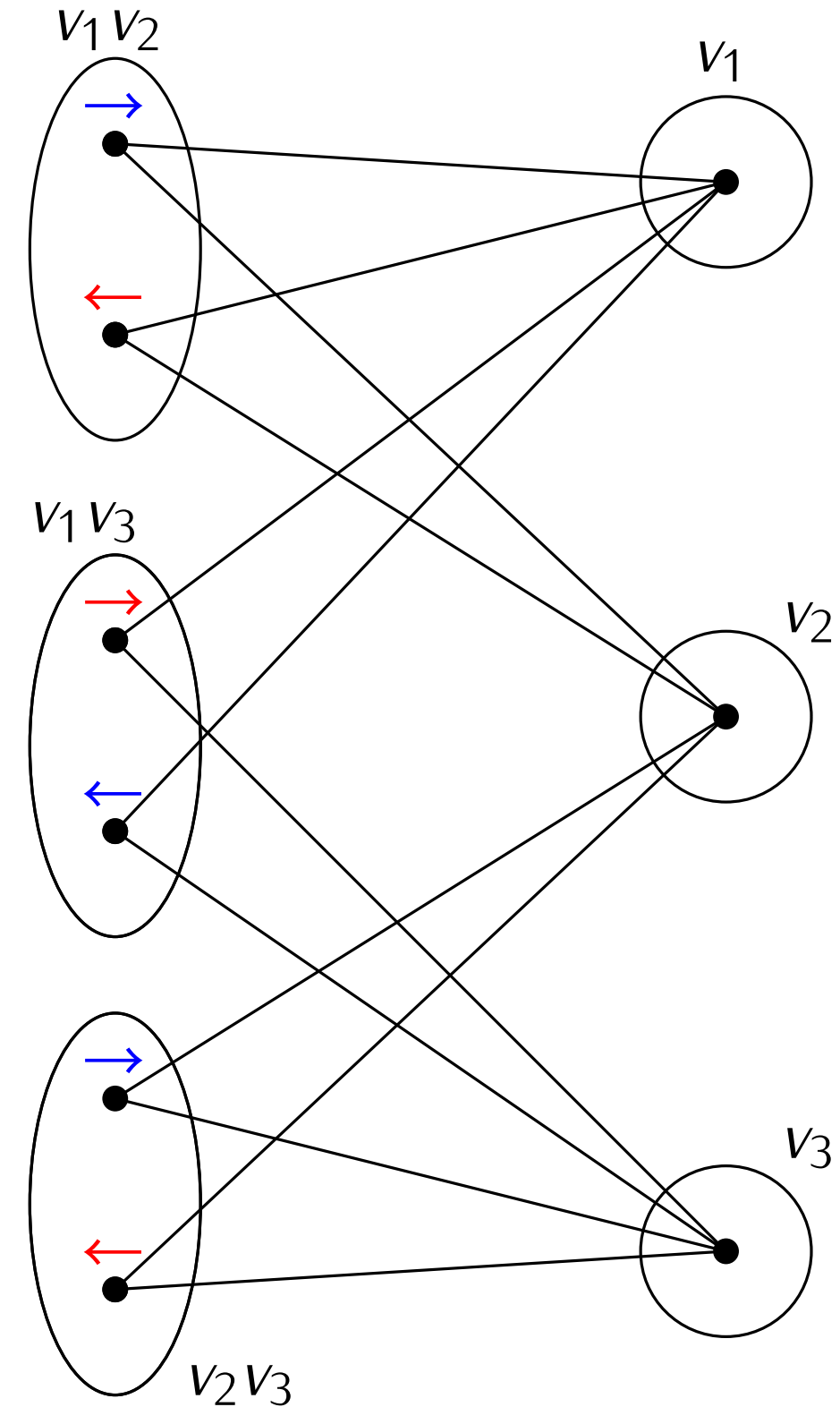
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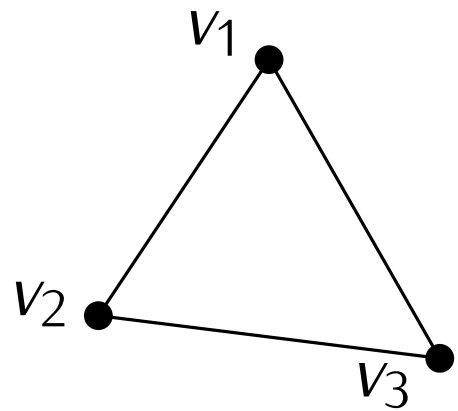


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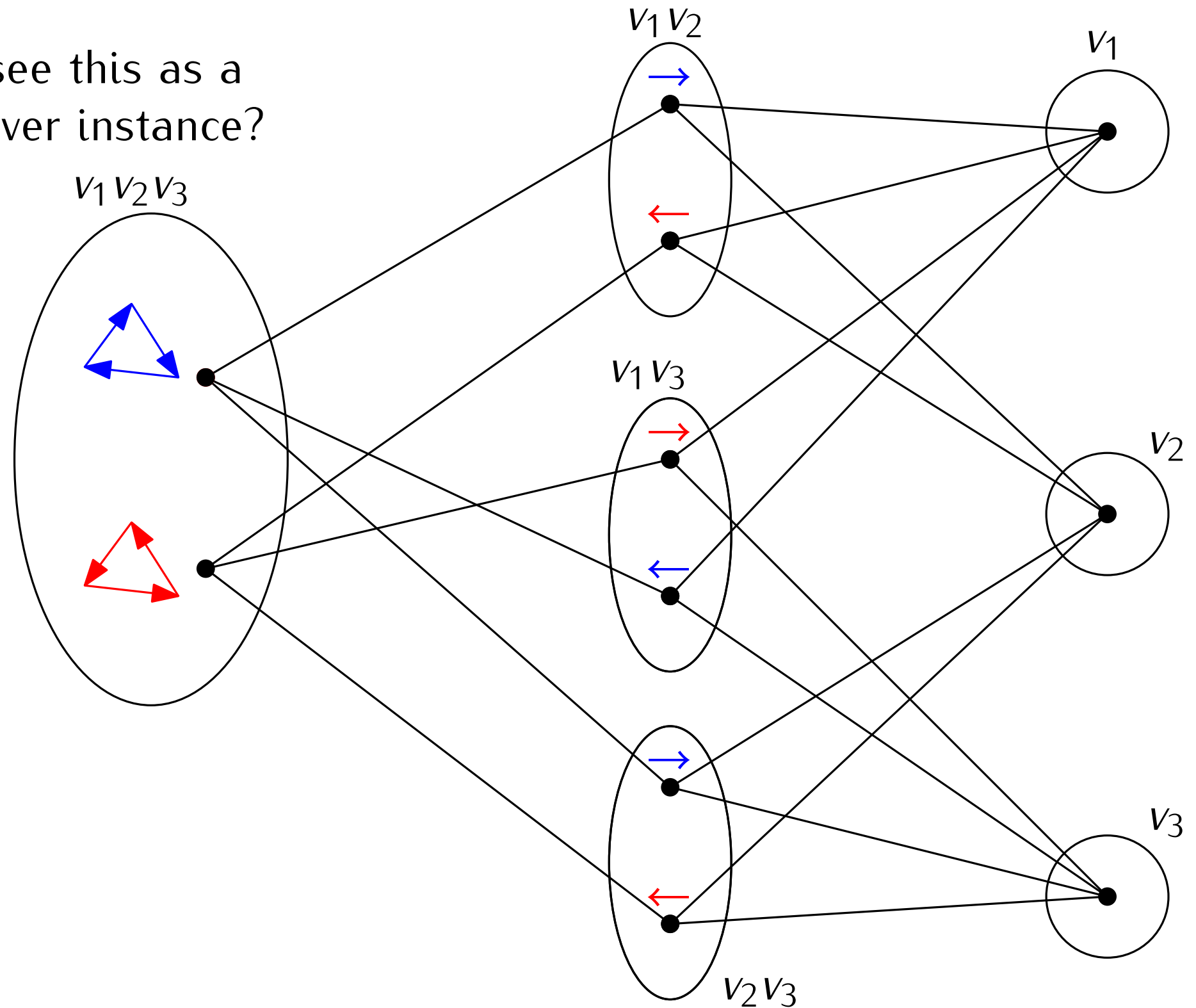


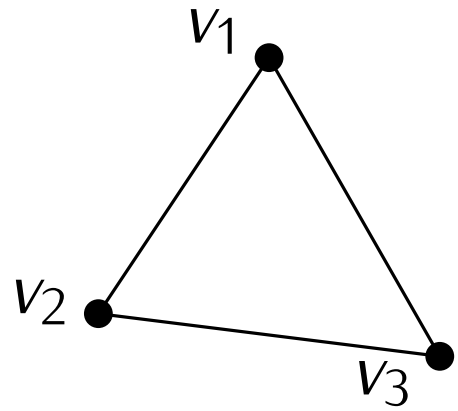
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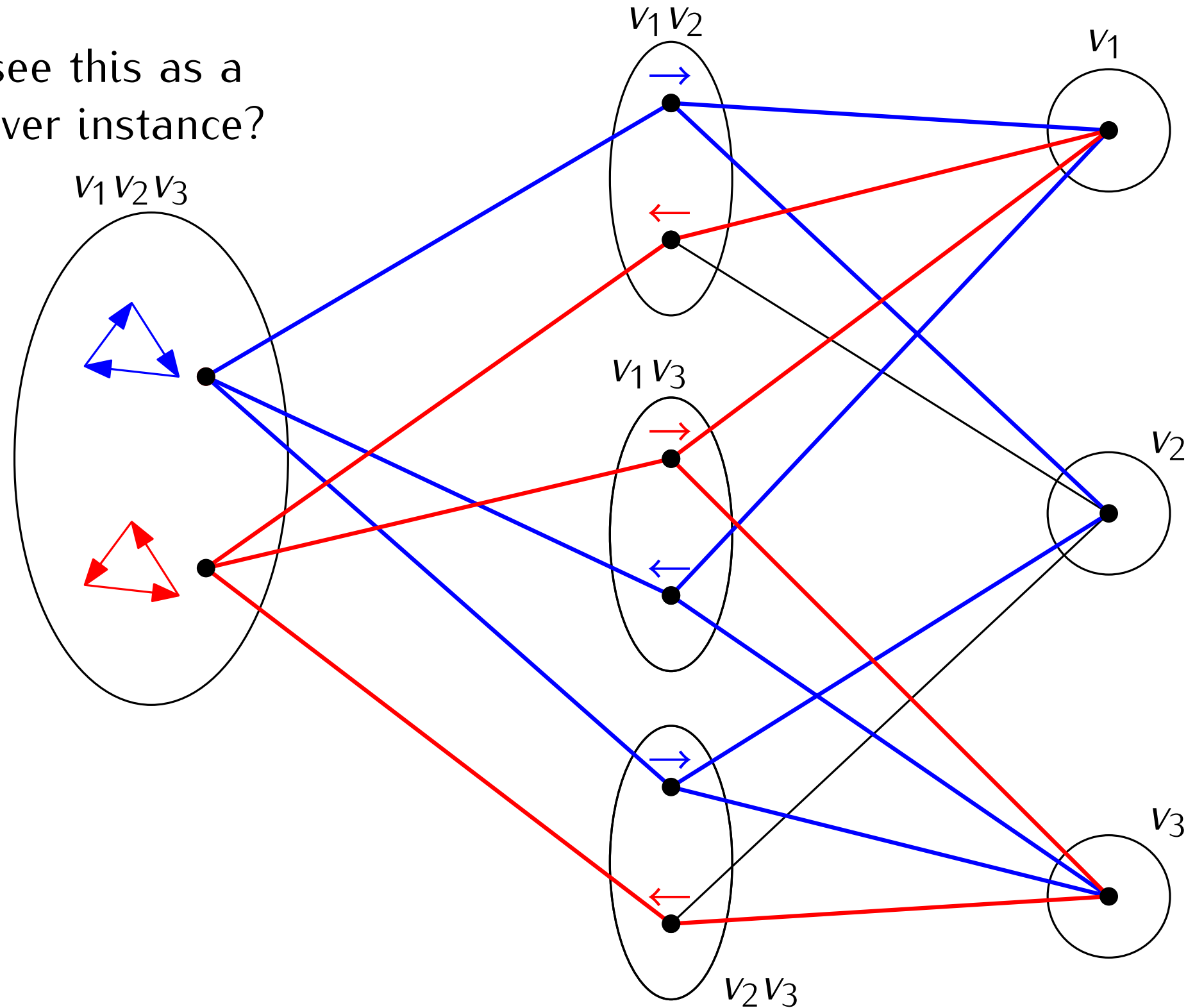


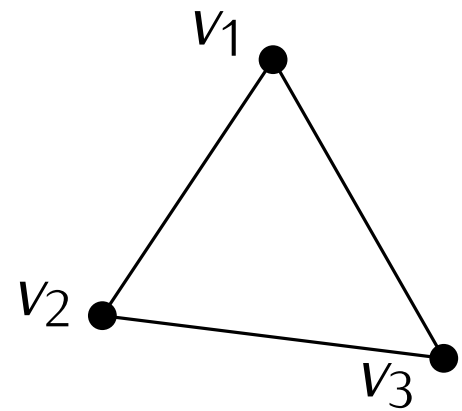
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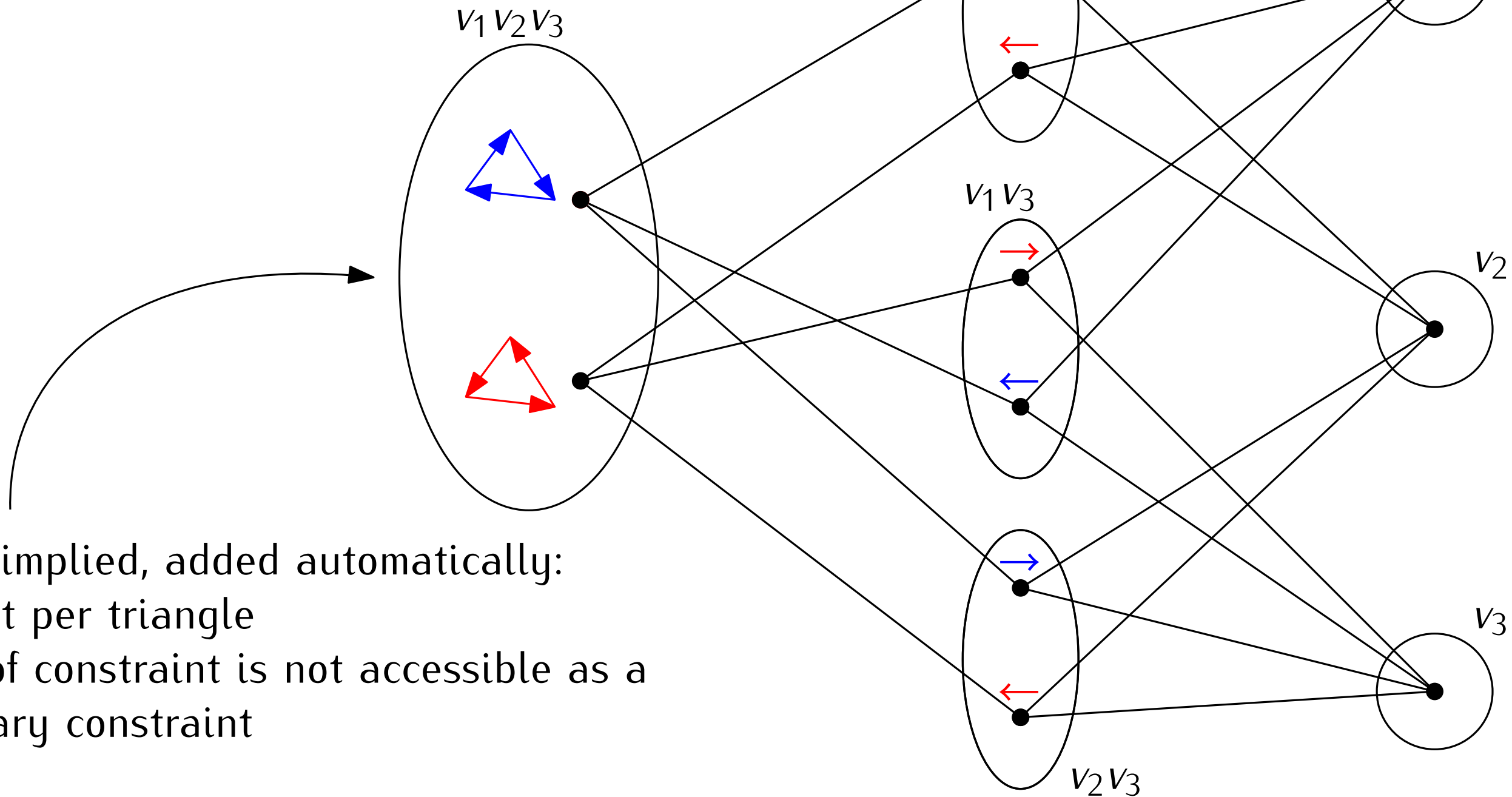


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This potato is implied, added automatically:

- 1 constraint per triangle
- This type of constraint is not accessible as a “real” ternary constraint

All problems so far:

- reduce to label cover over a finite set
- some **structural** constraints are "automatically" added
- structural constraints are not part of the "normal" constraints
- scope of structural constraints is of **bounded** size
- Except for 3-coloring, not $\text{CSP}(\mathbb{A})$ for a finite \mathbb{A} , yet $\text{CSP}(\mathbb{A})$ for some structures (disjoint unions)

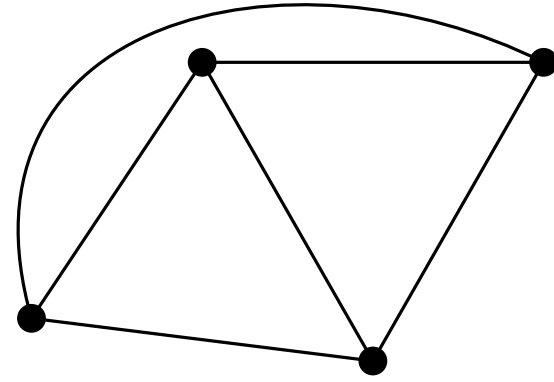
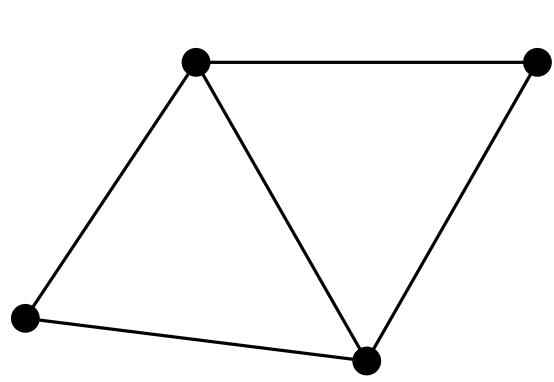
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What templates to study these problems?

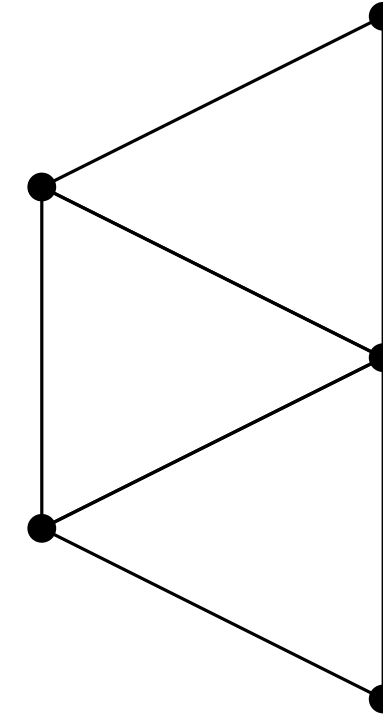
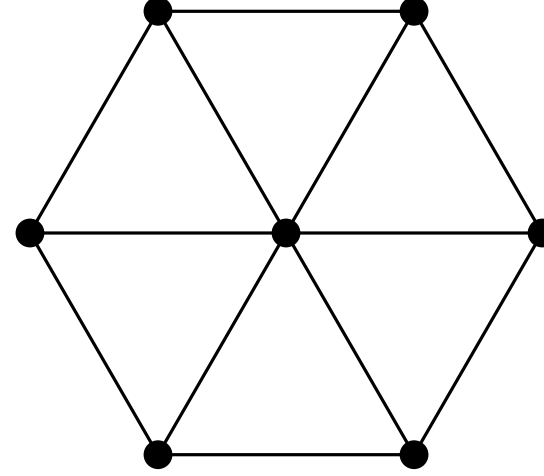
How to study the complexity of the problems using these templates?

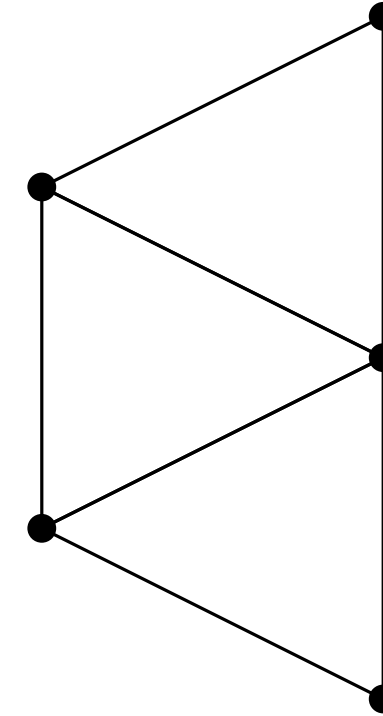
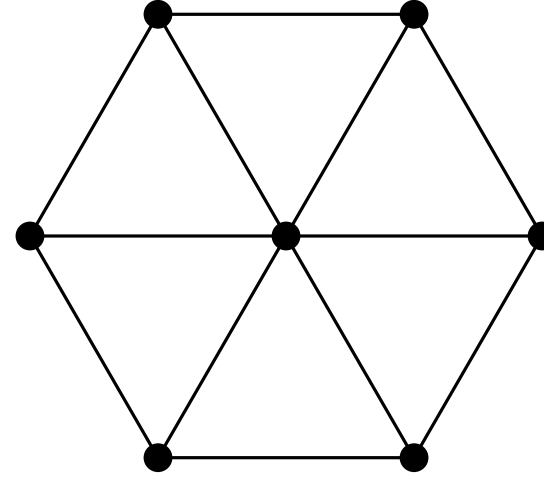
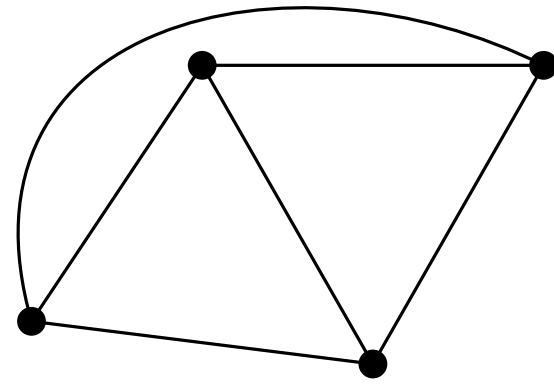
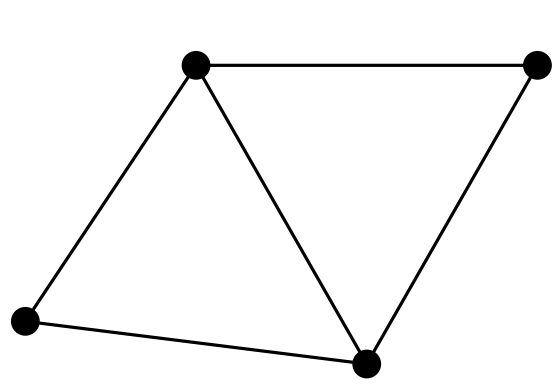
Orbit-finiteness,
reducts of finitely bounded homogeneous structures



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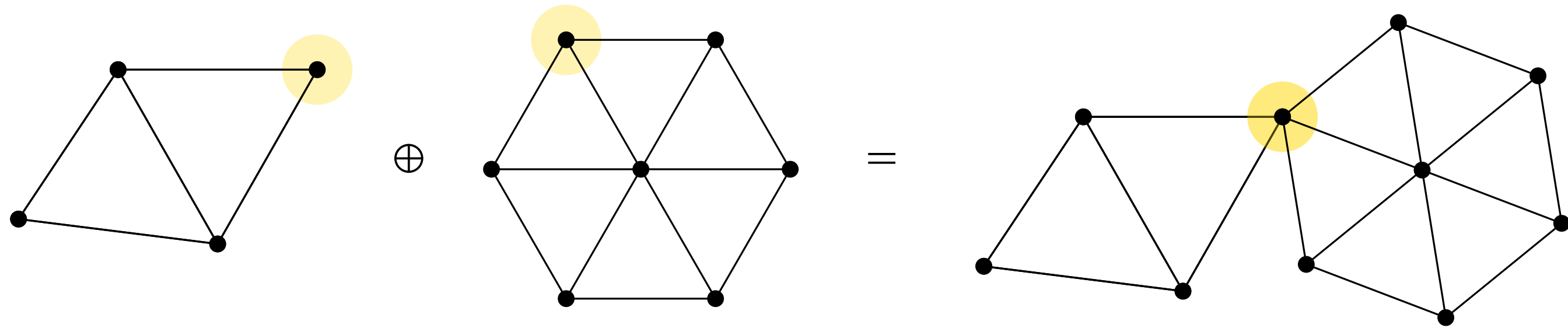
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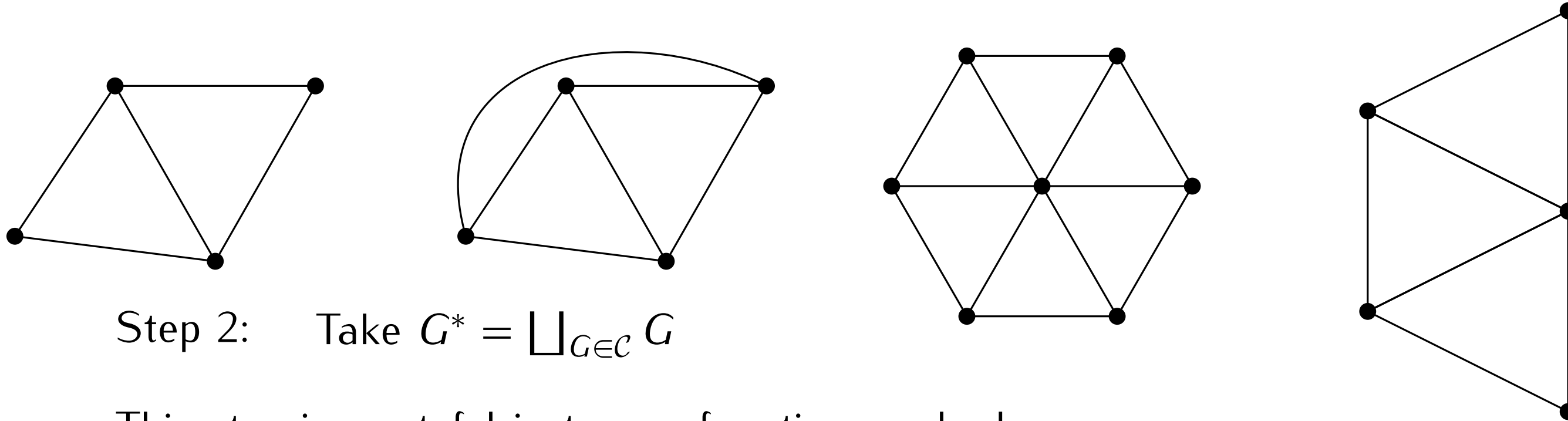




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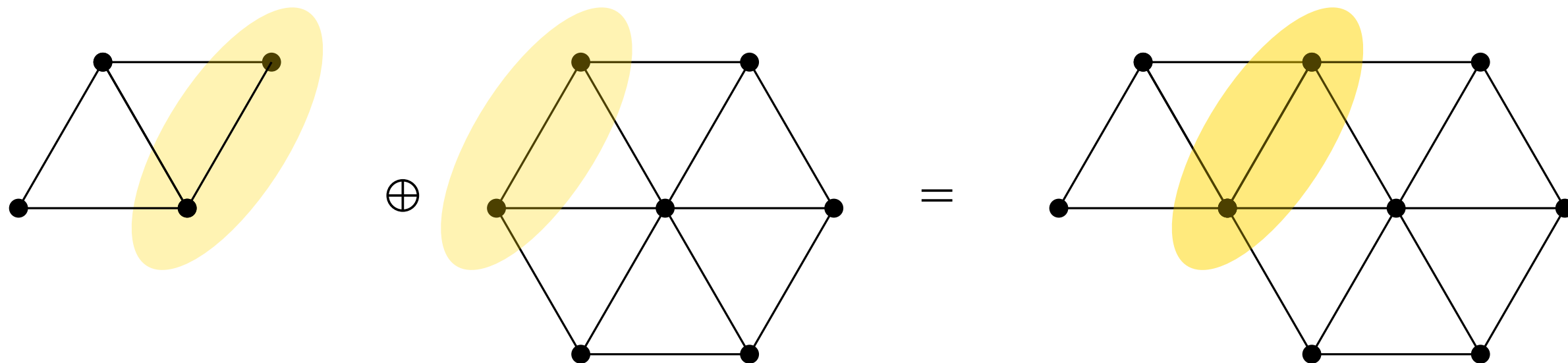
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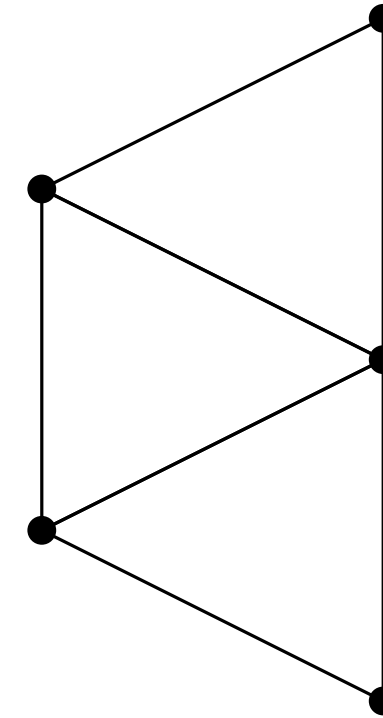
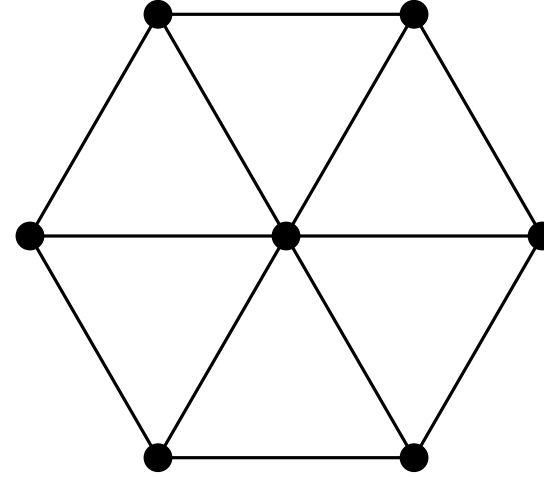
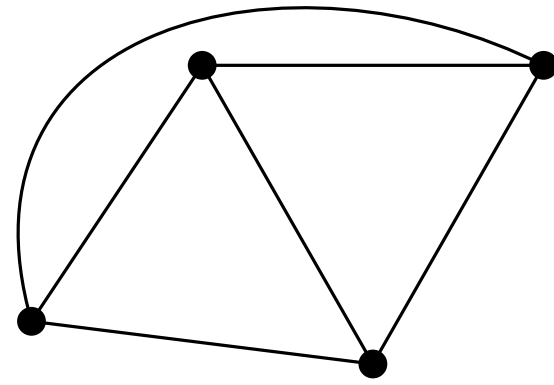
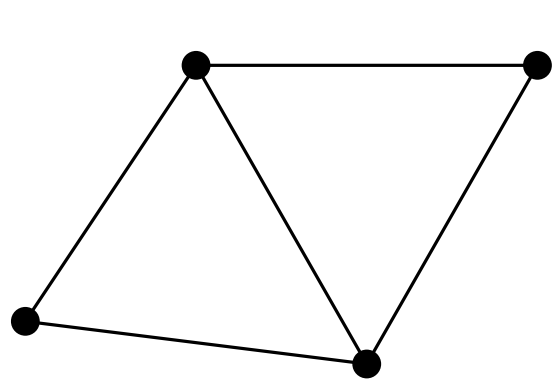




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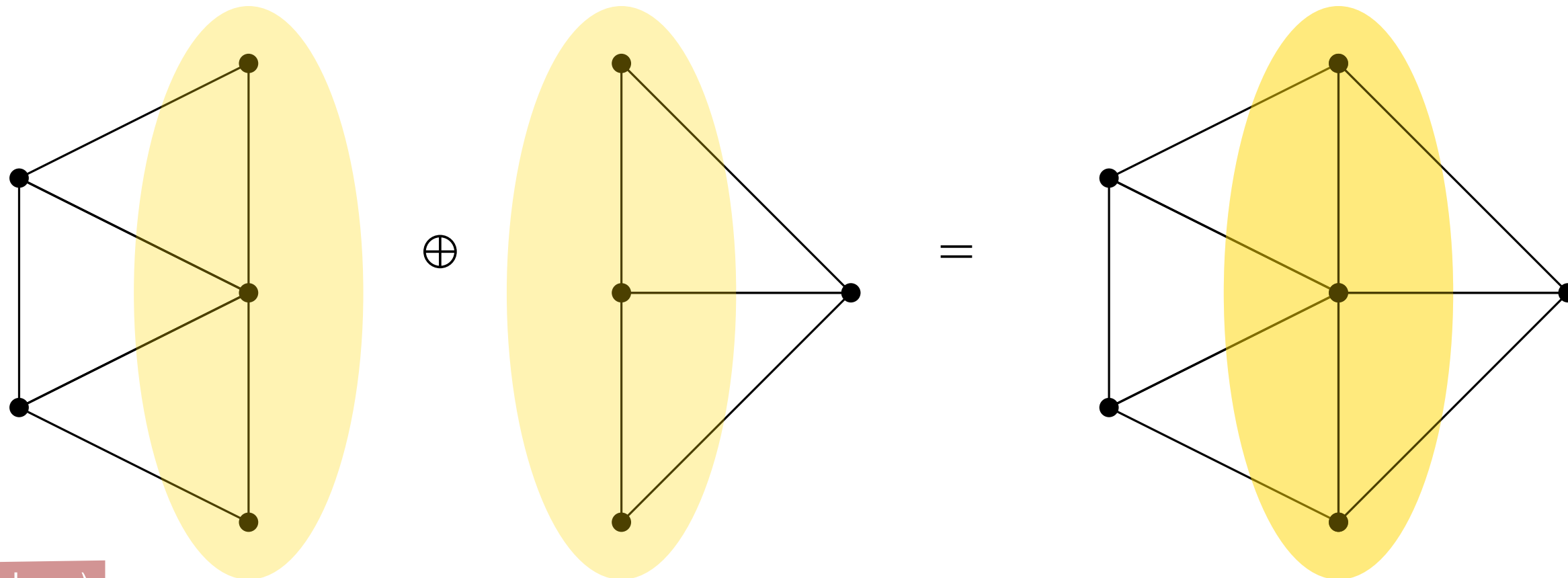
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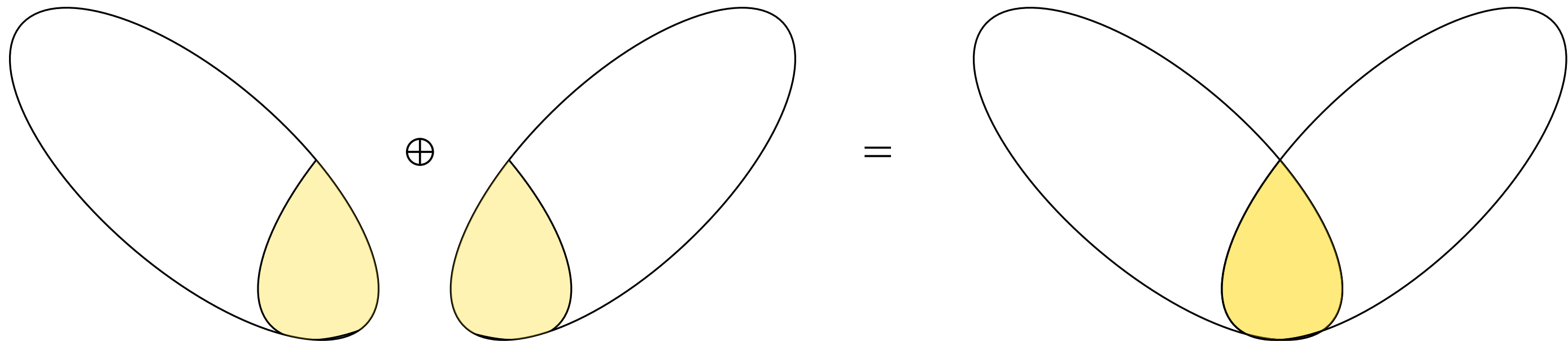
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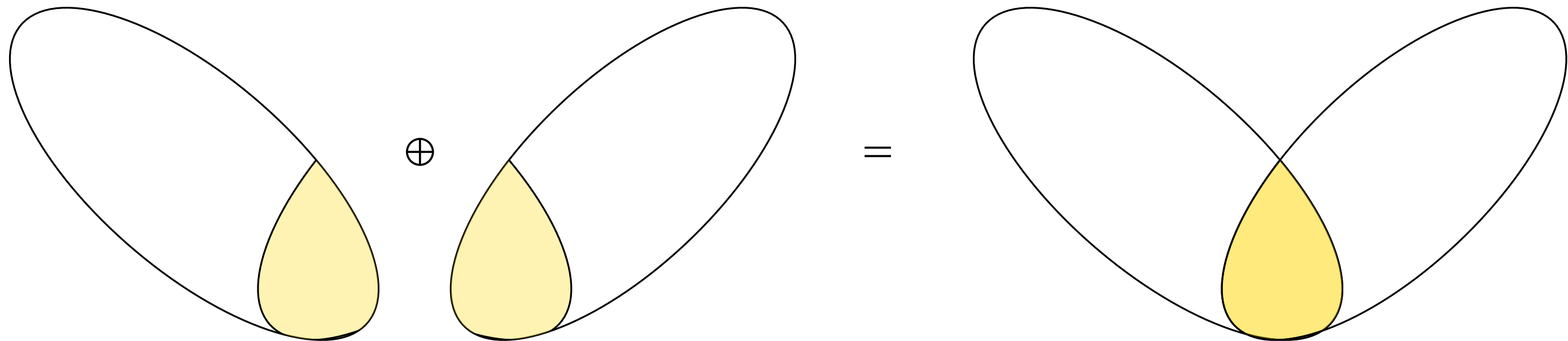


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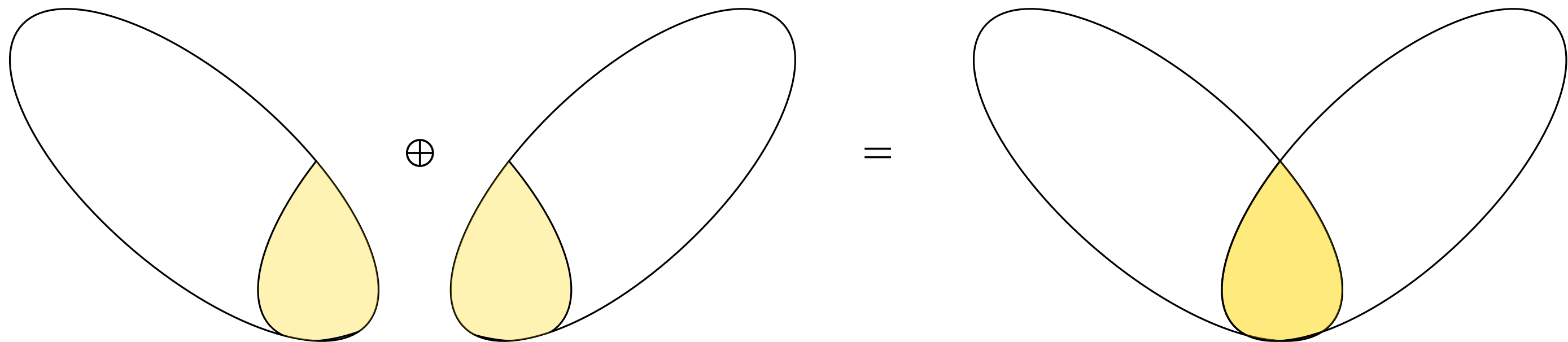
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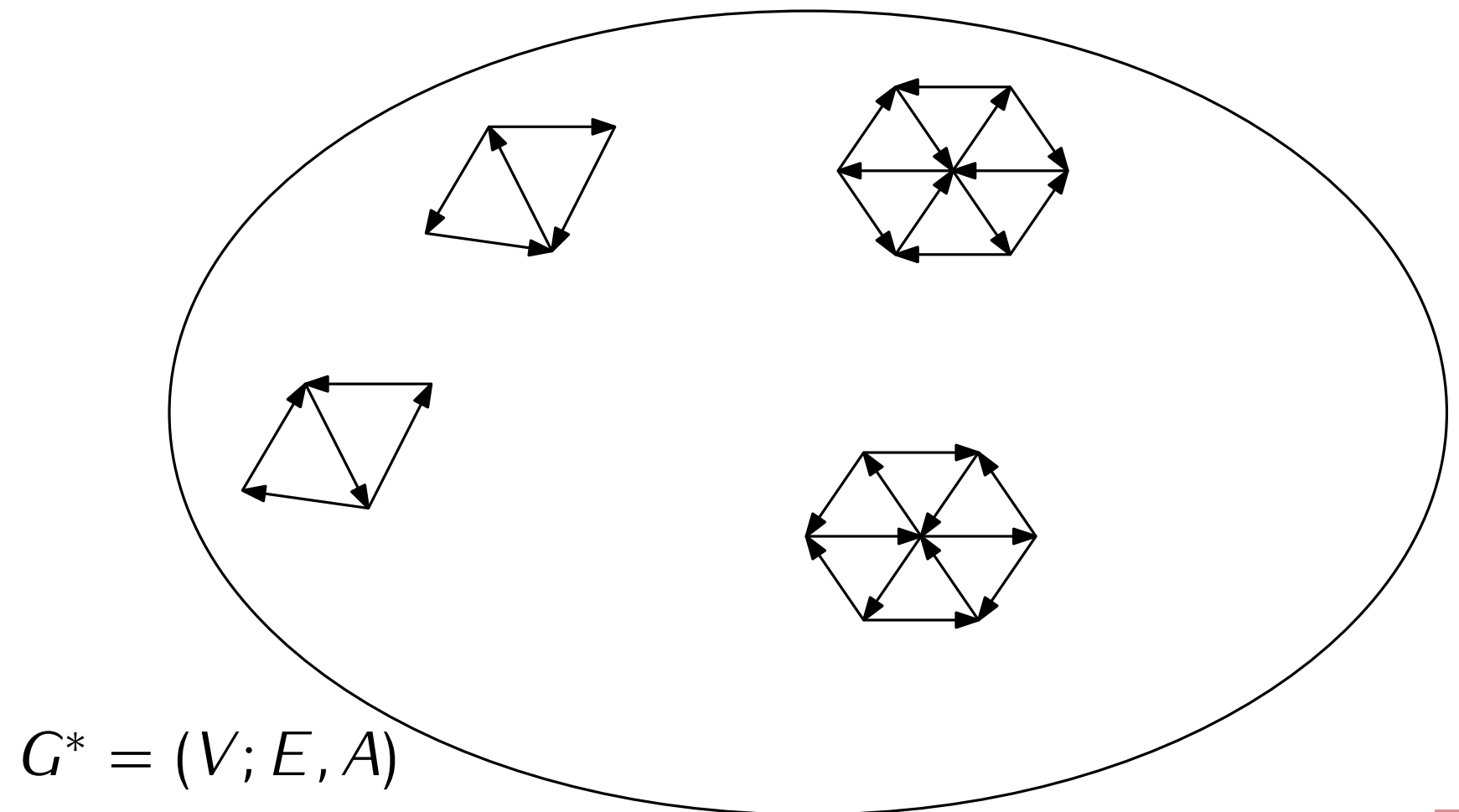
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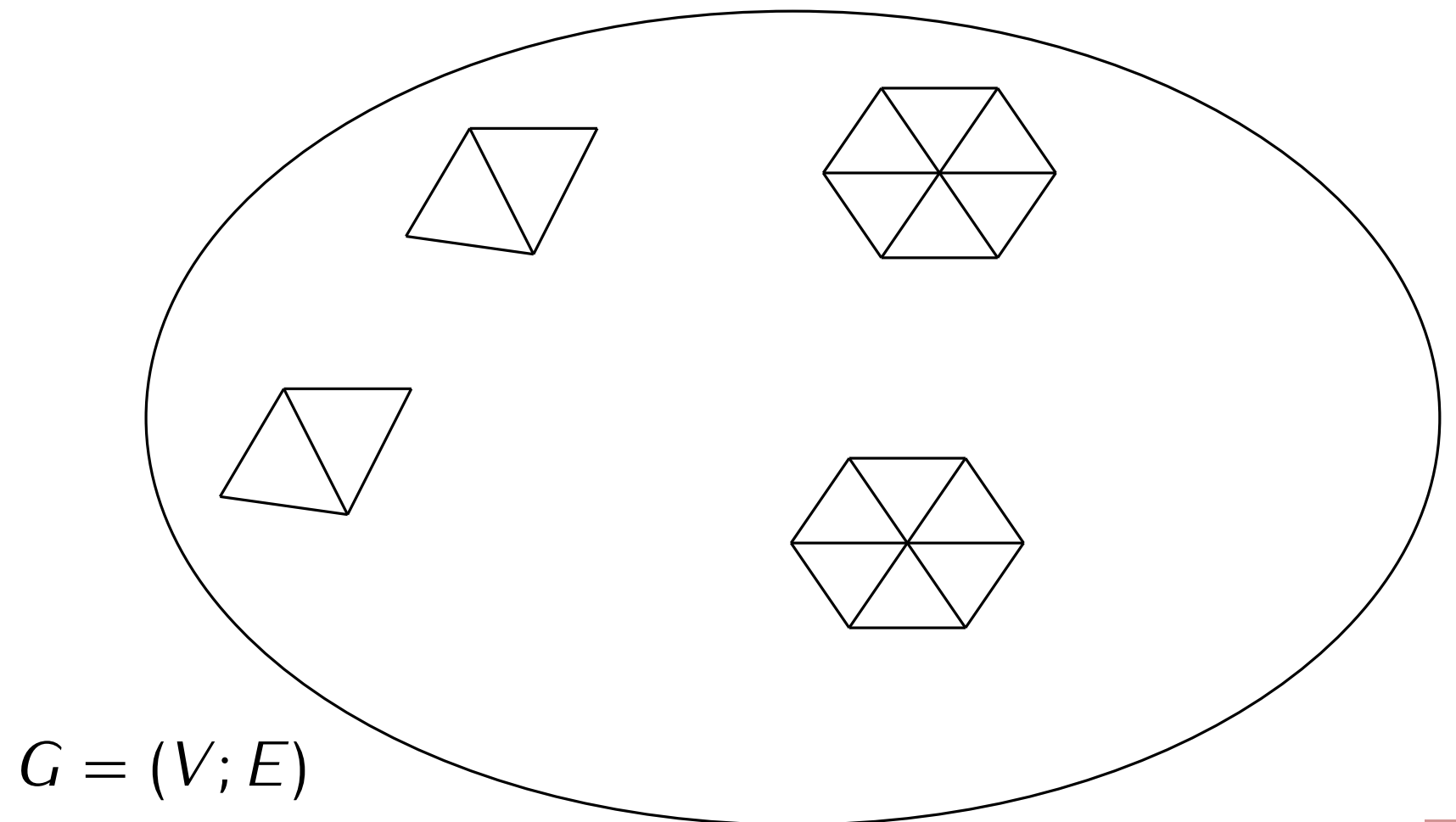
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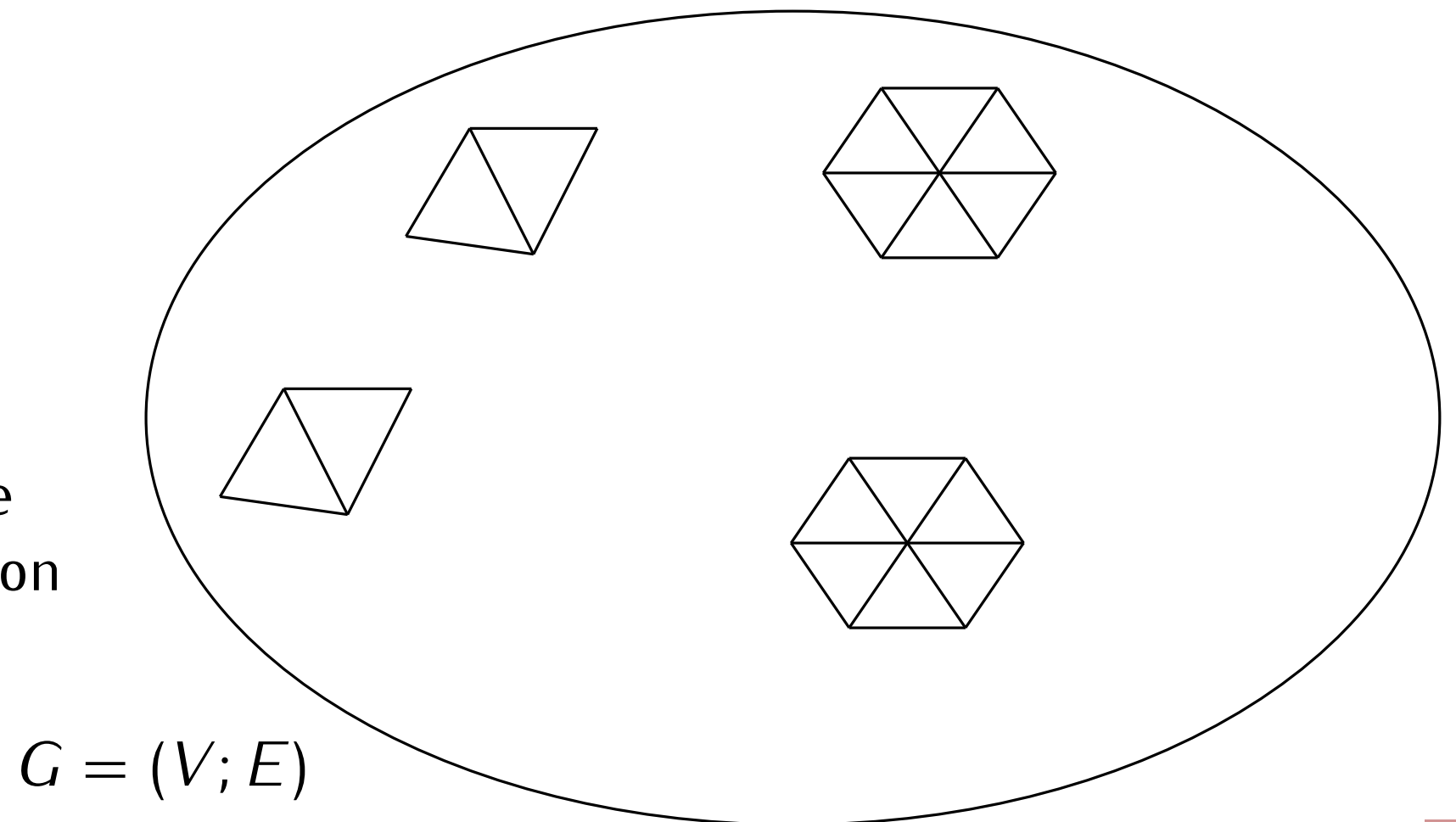
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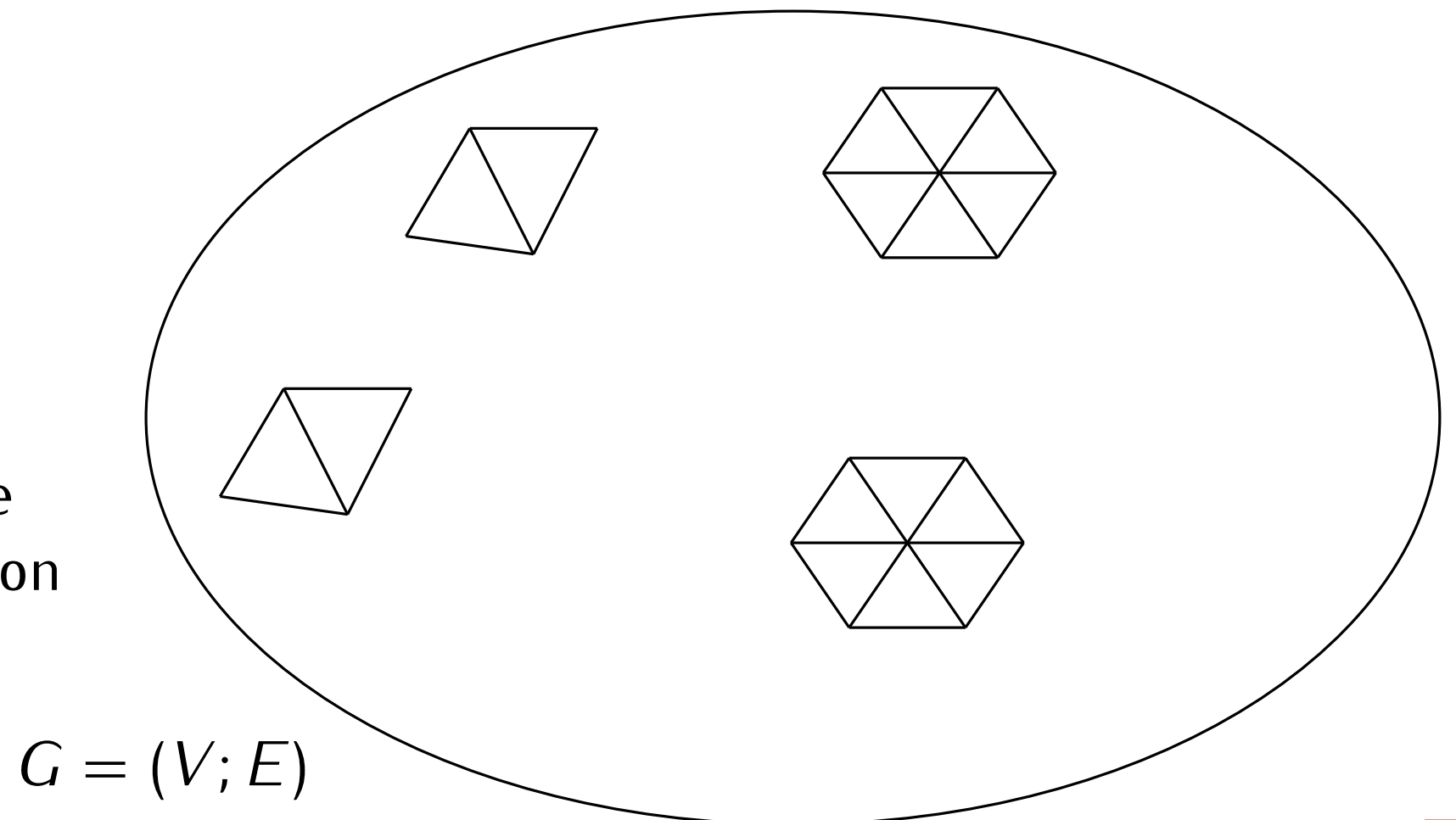
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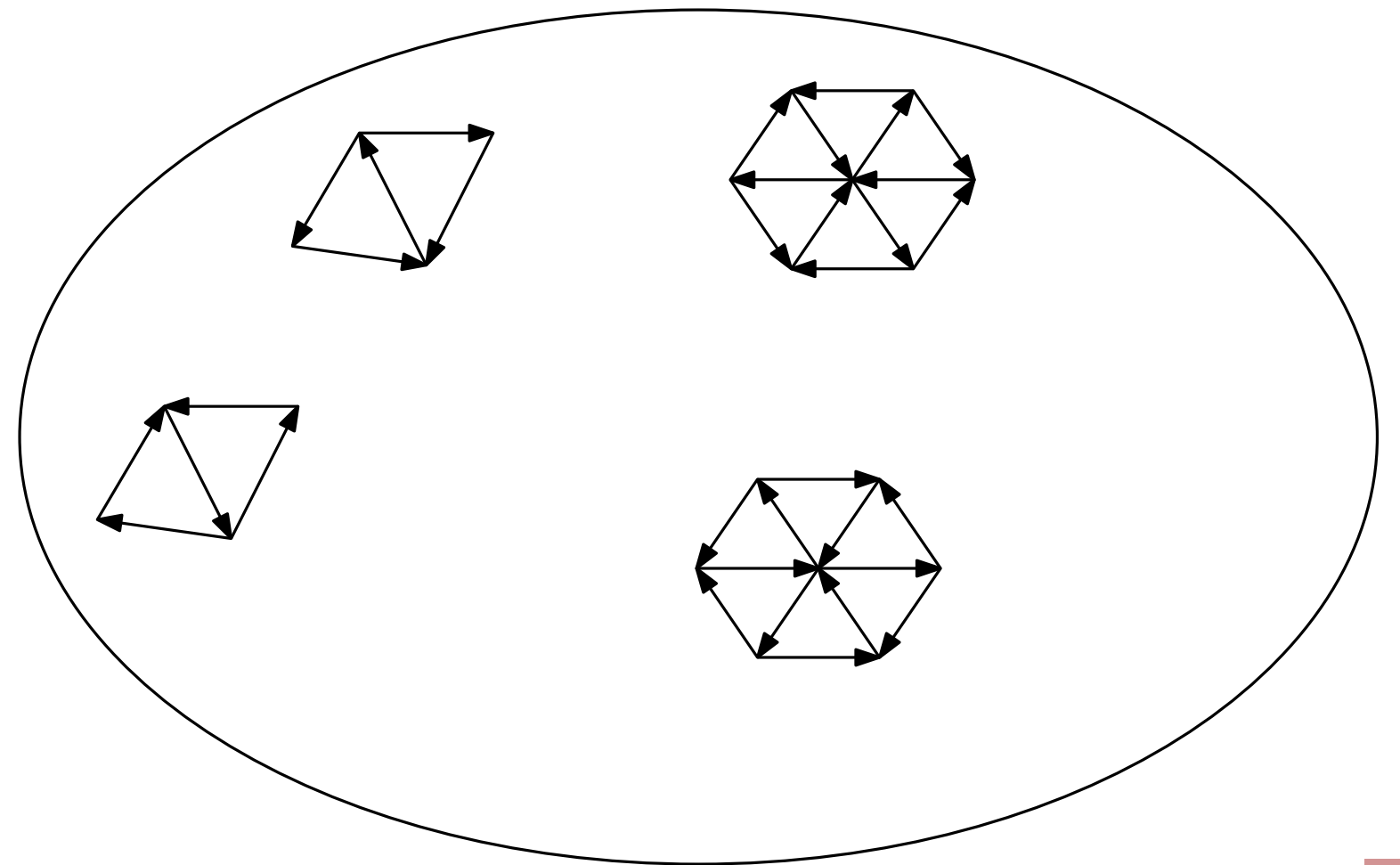
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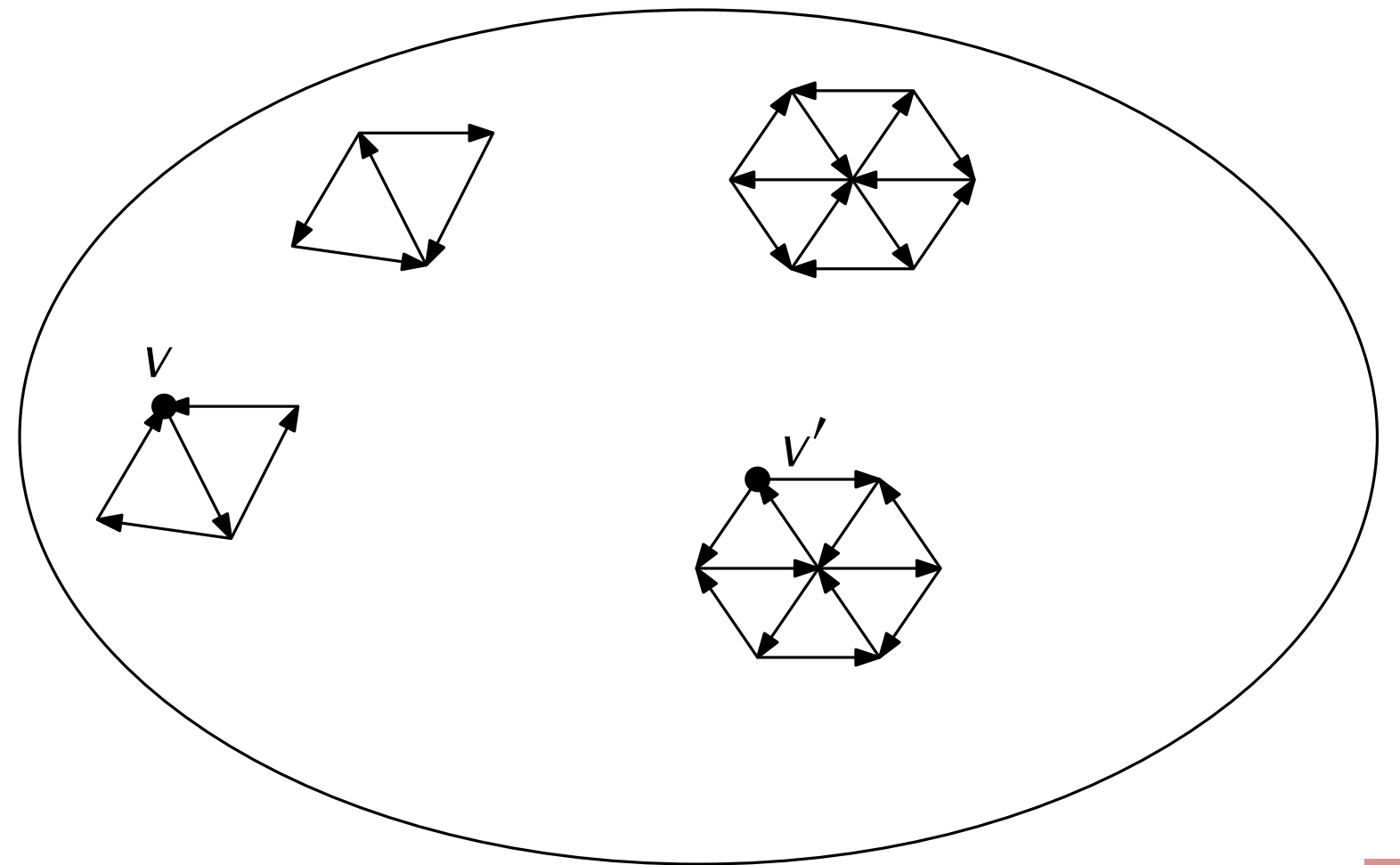
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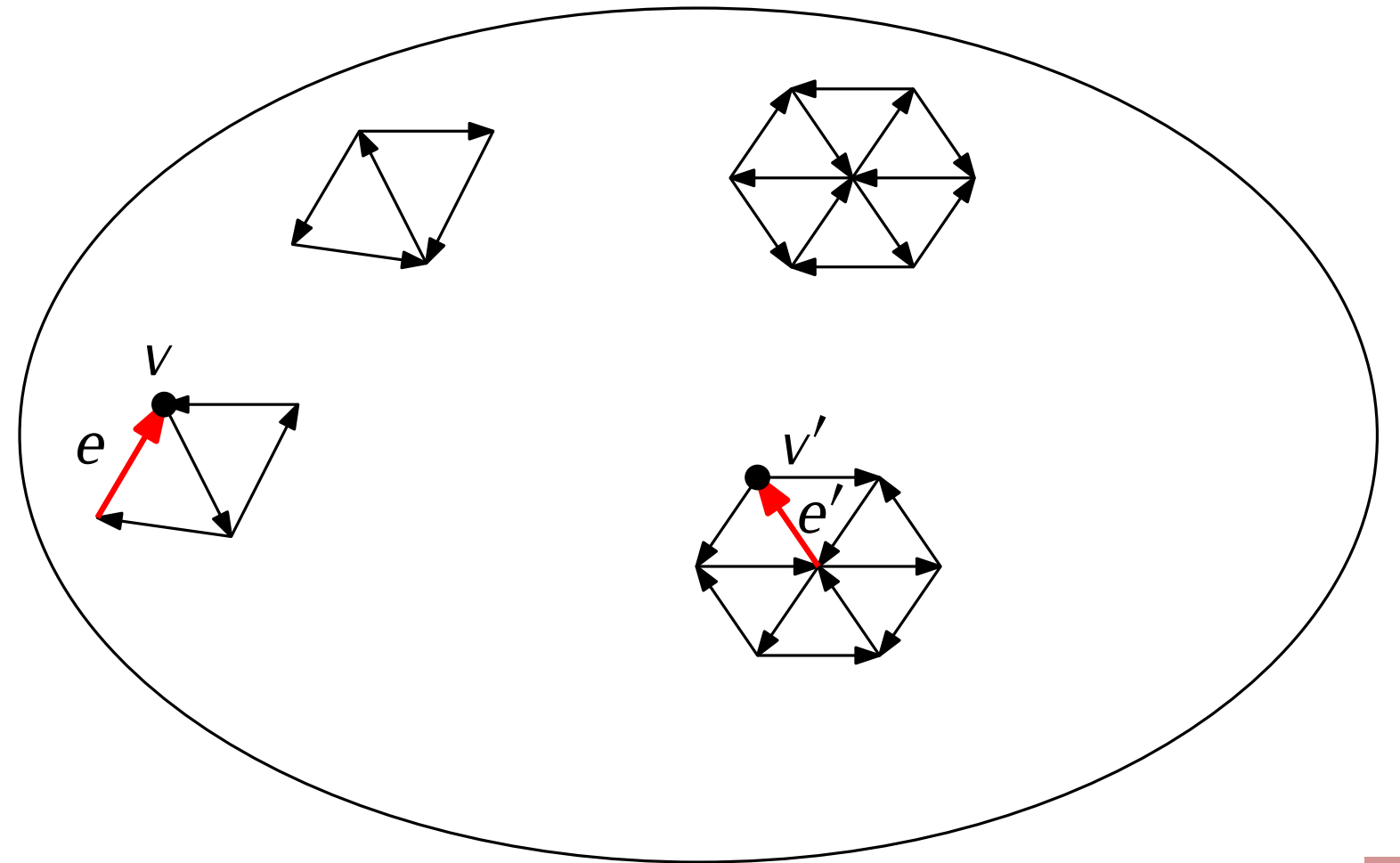


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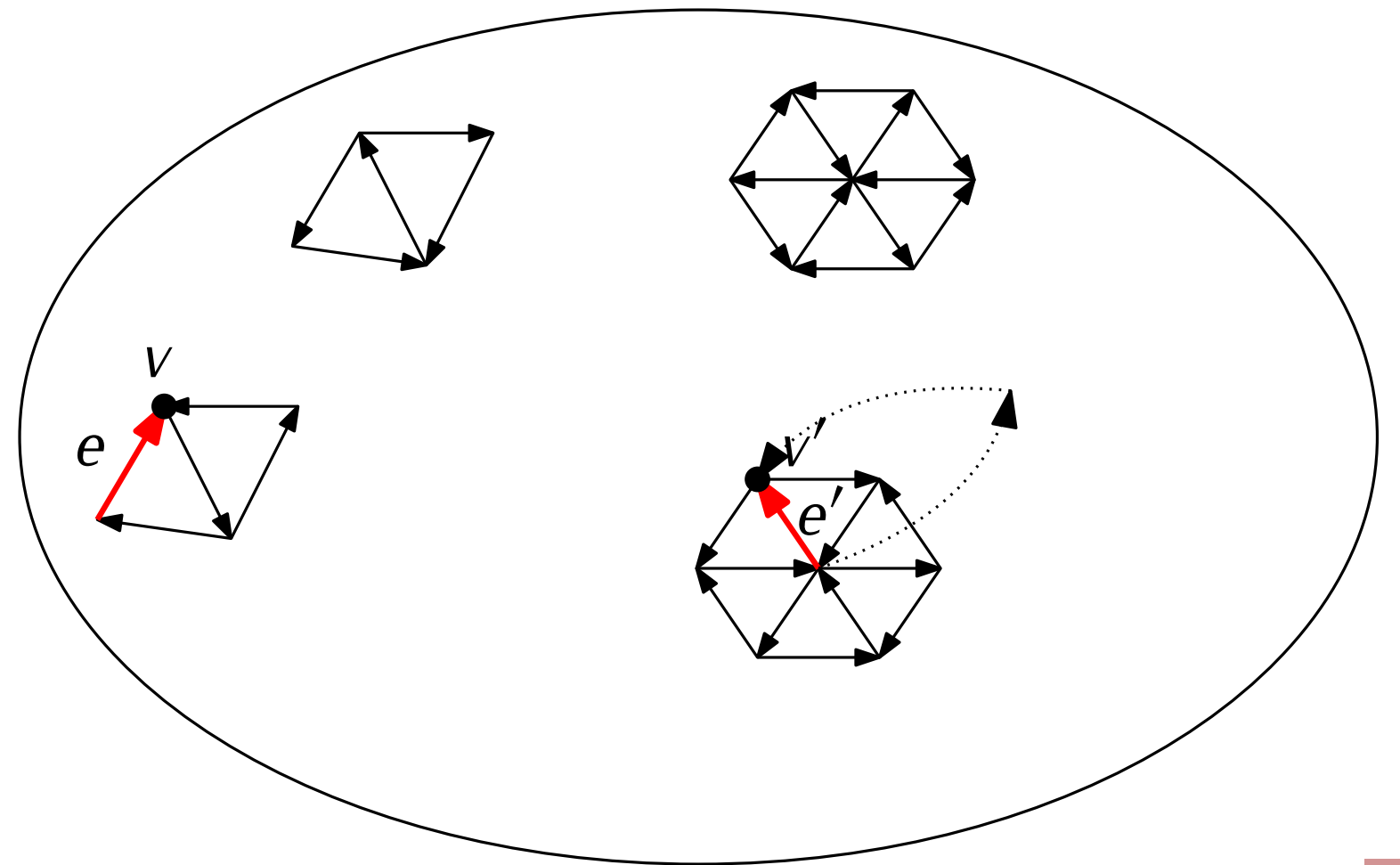


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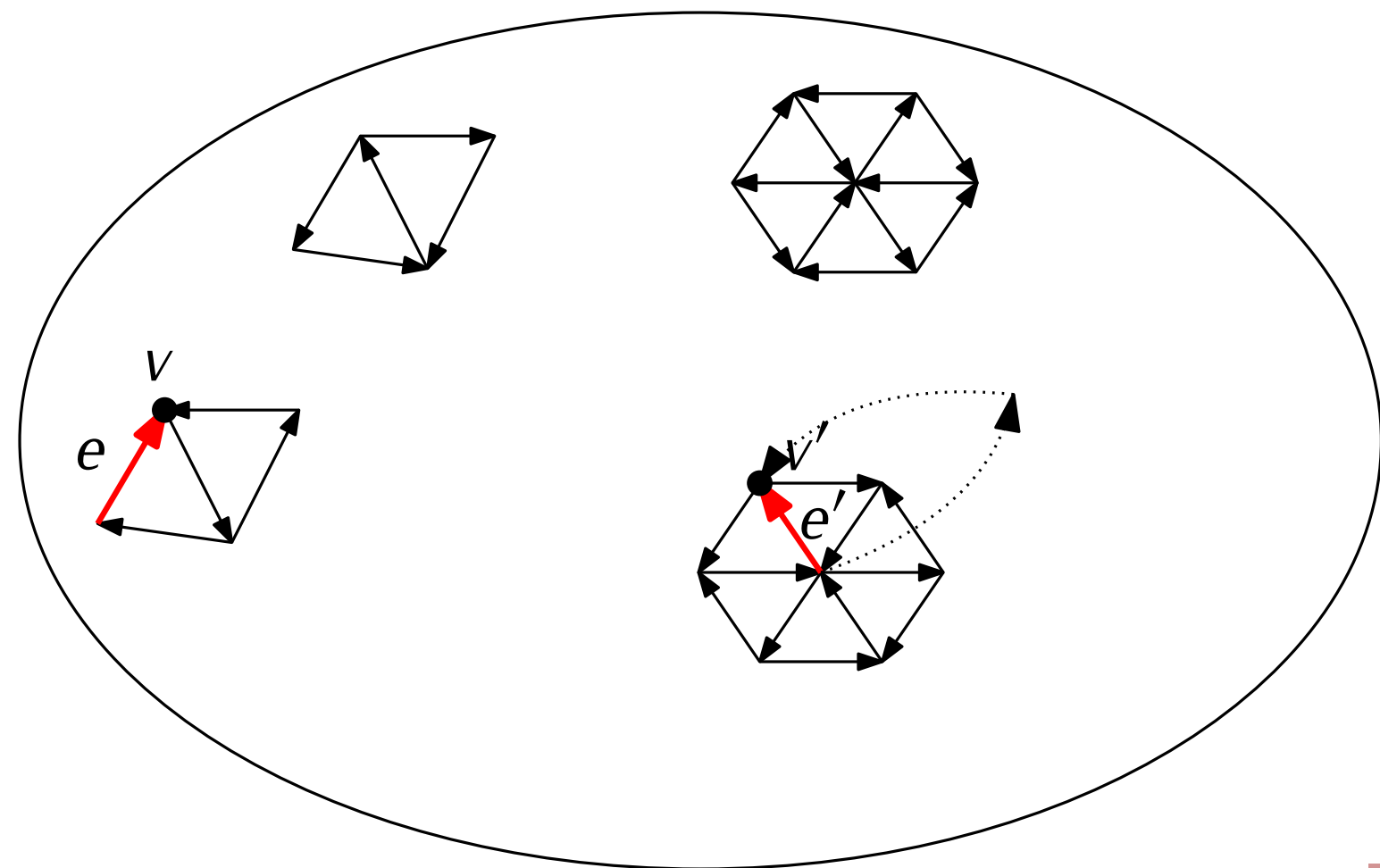
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Finitely many directed graphs on n vertices

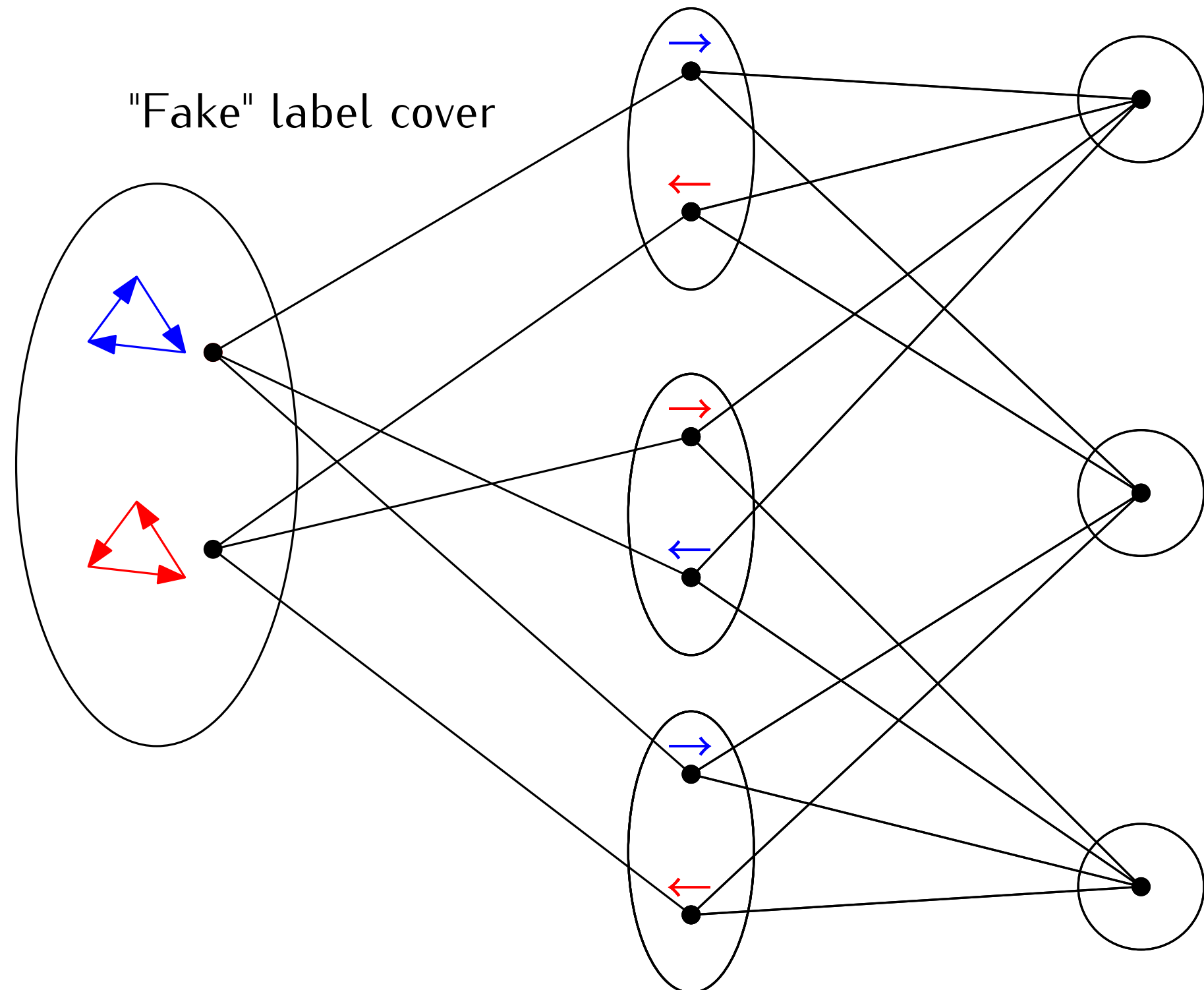
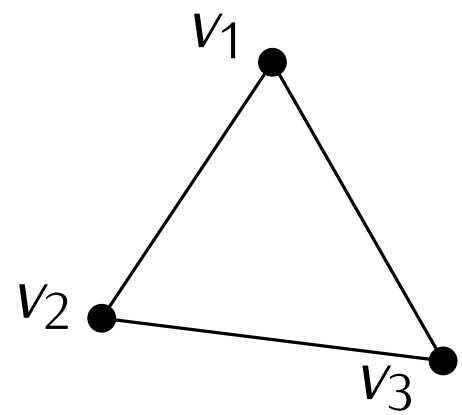
$\rightsquigarrow \text{Aut}(G^*) \curvearrowright V^n$ has finitely many orbits

$\text{Aut}(G)$ is **oligomorphic** / G is **ω -categorical**



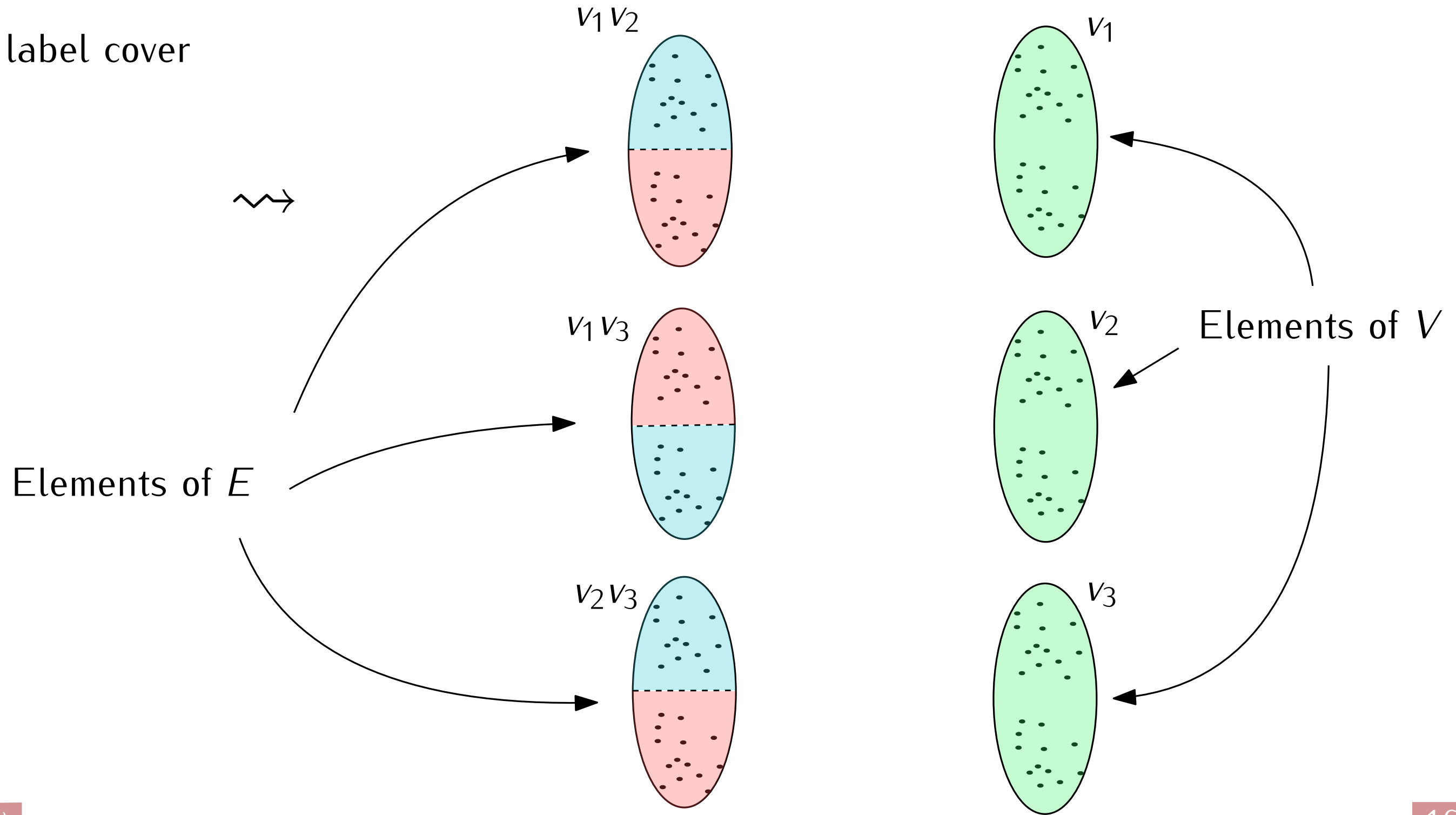
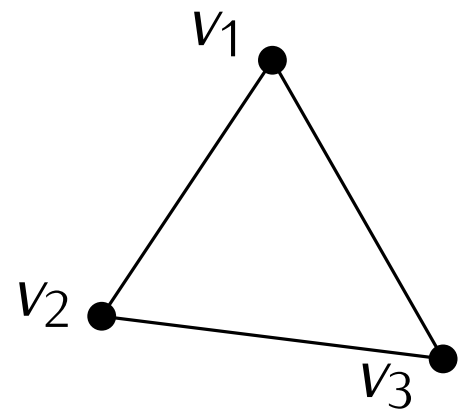
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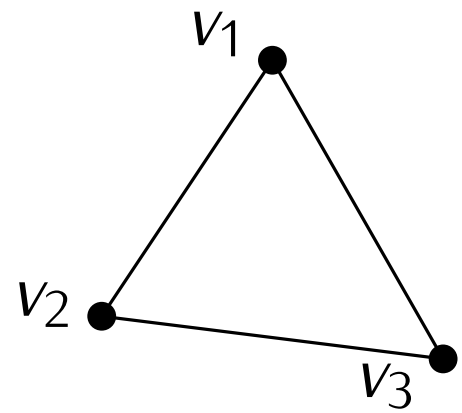
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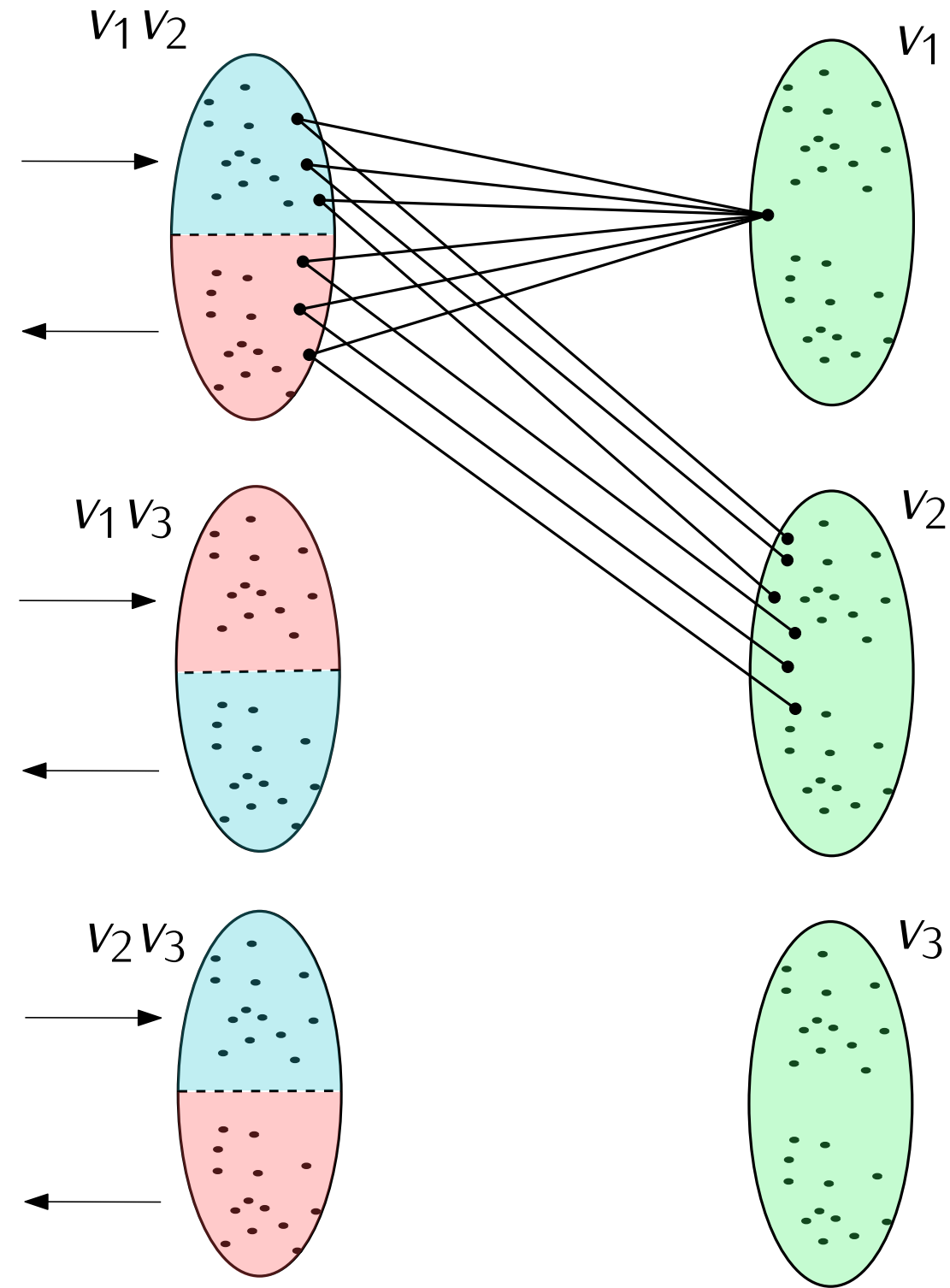


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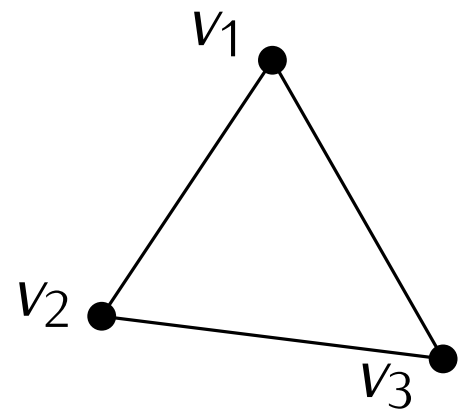


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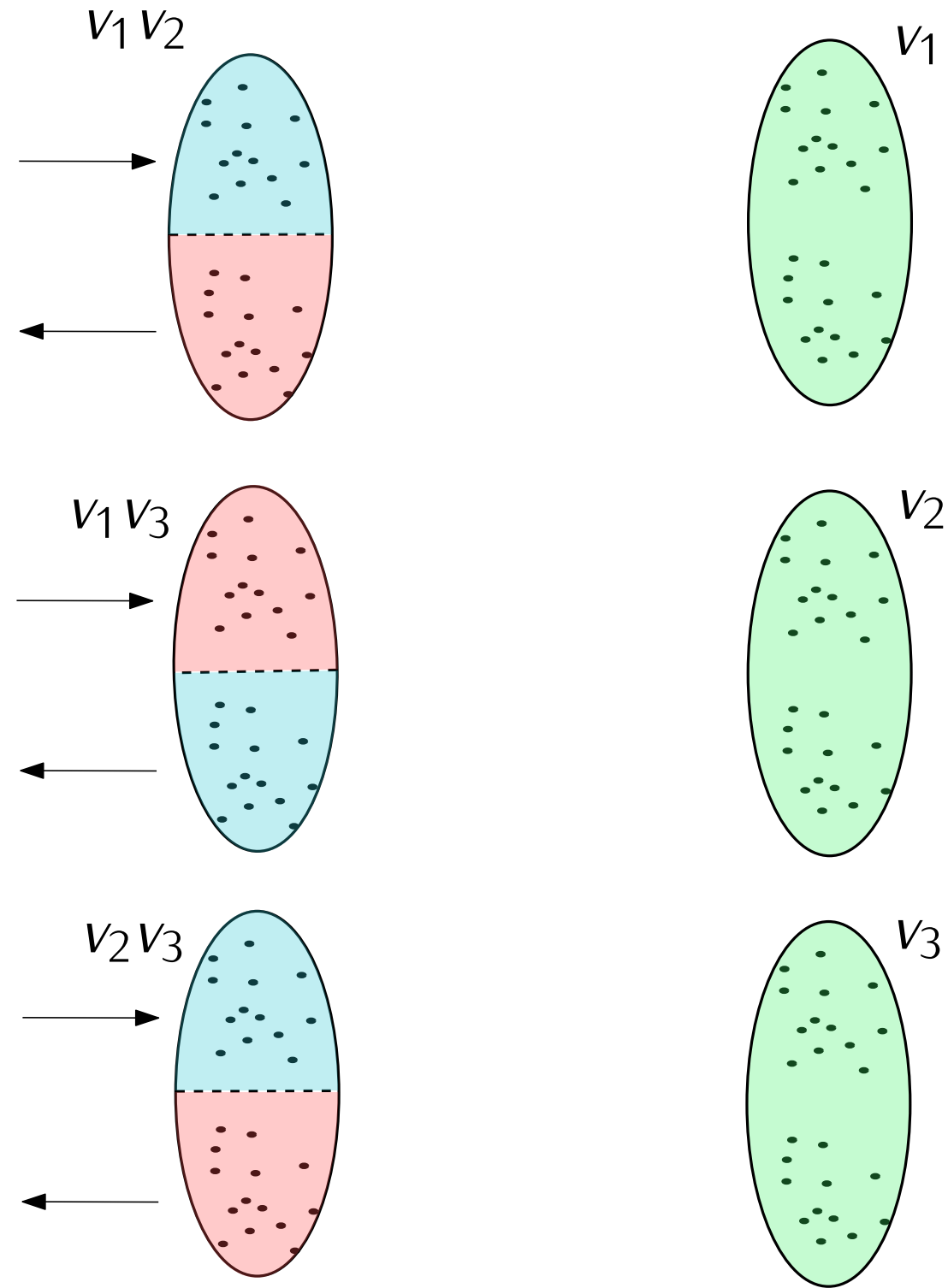
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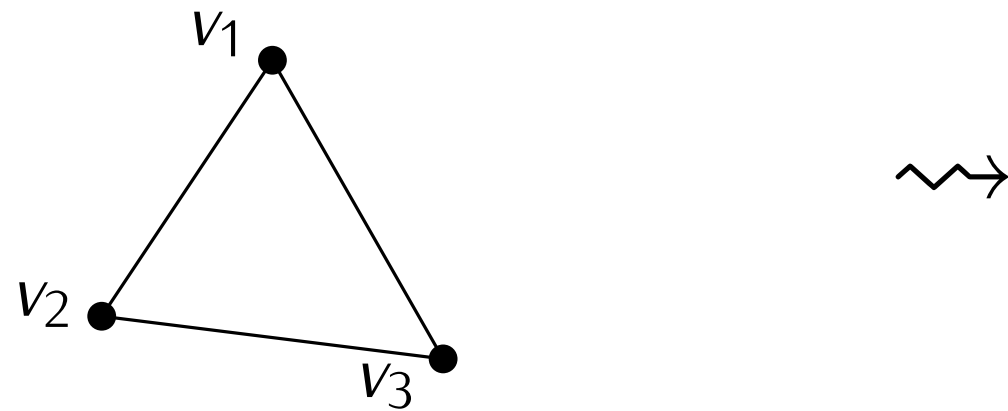
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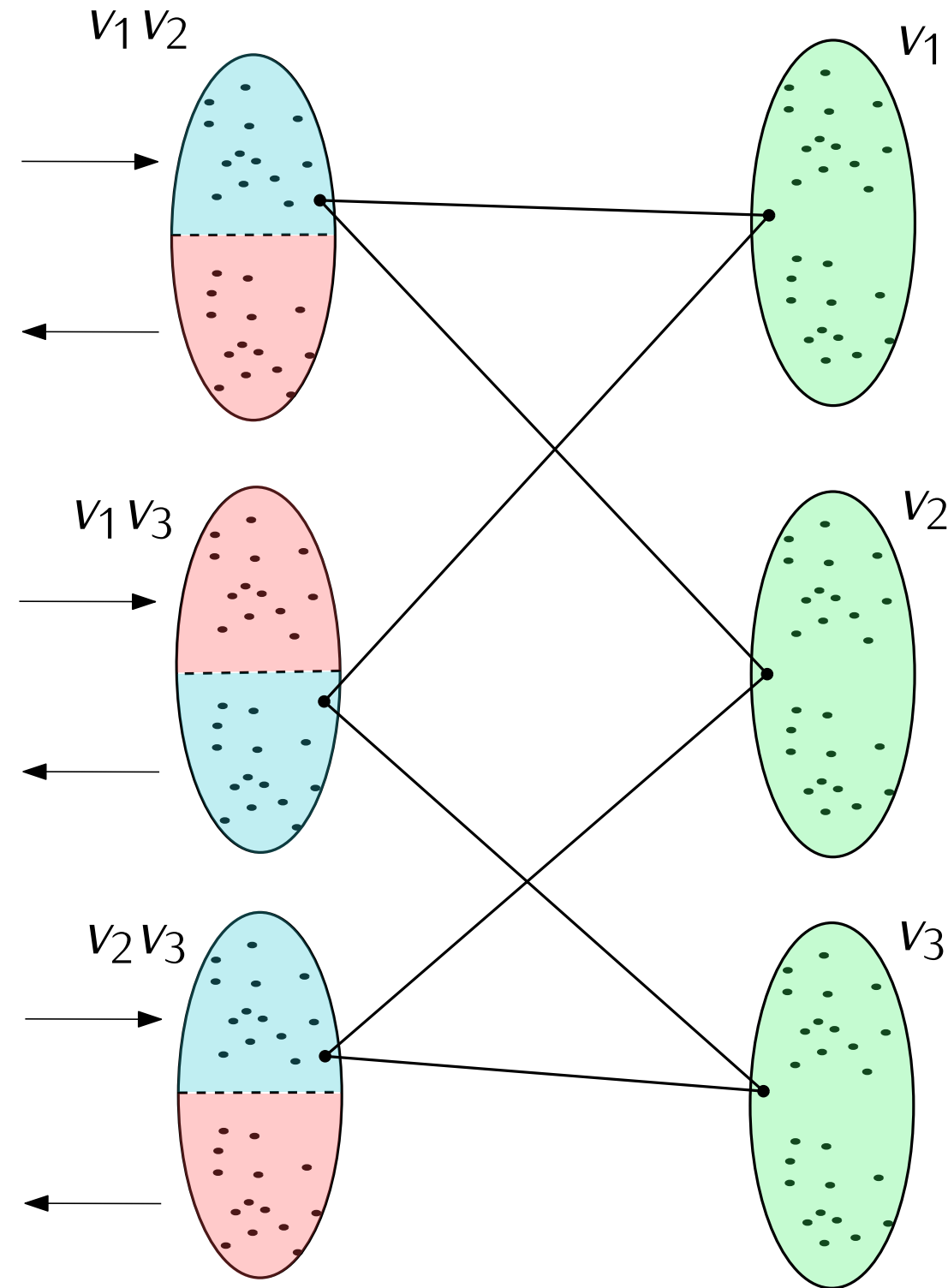
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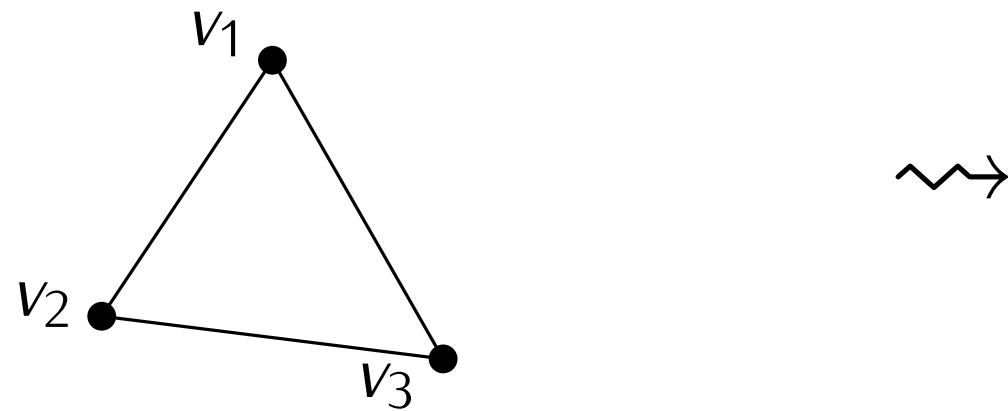
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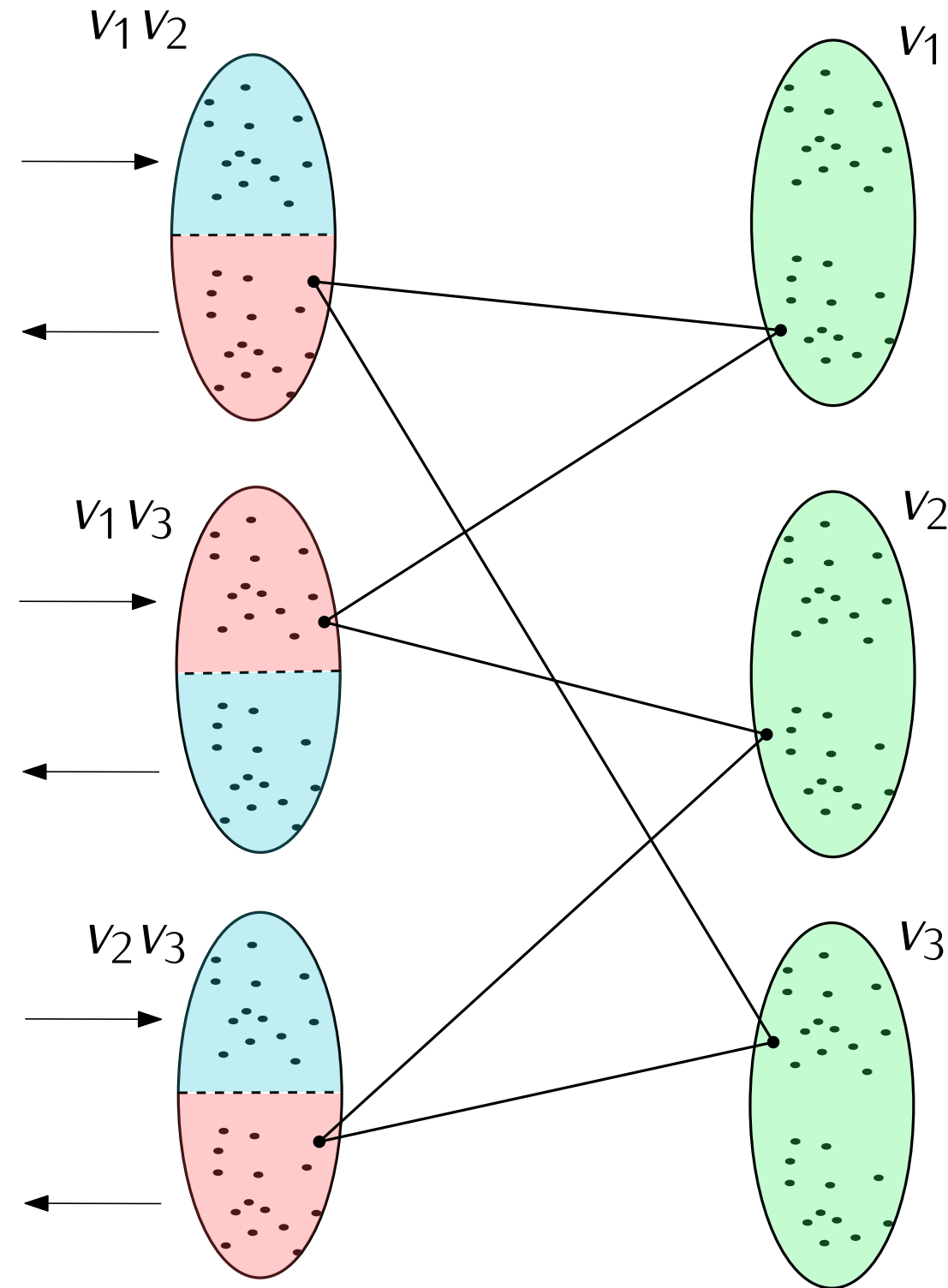


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Wrong because:

- \exists undecidable CSPs: polynomial equations over \mathbb{Z}
- \exists NP-intermediate CSPs if $P \neq NP$

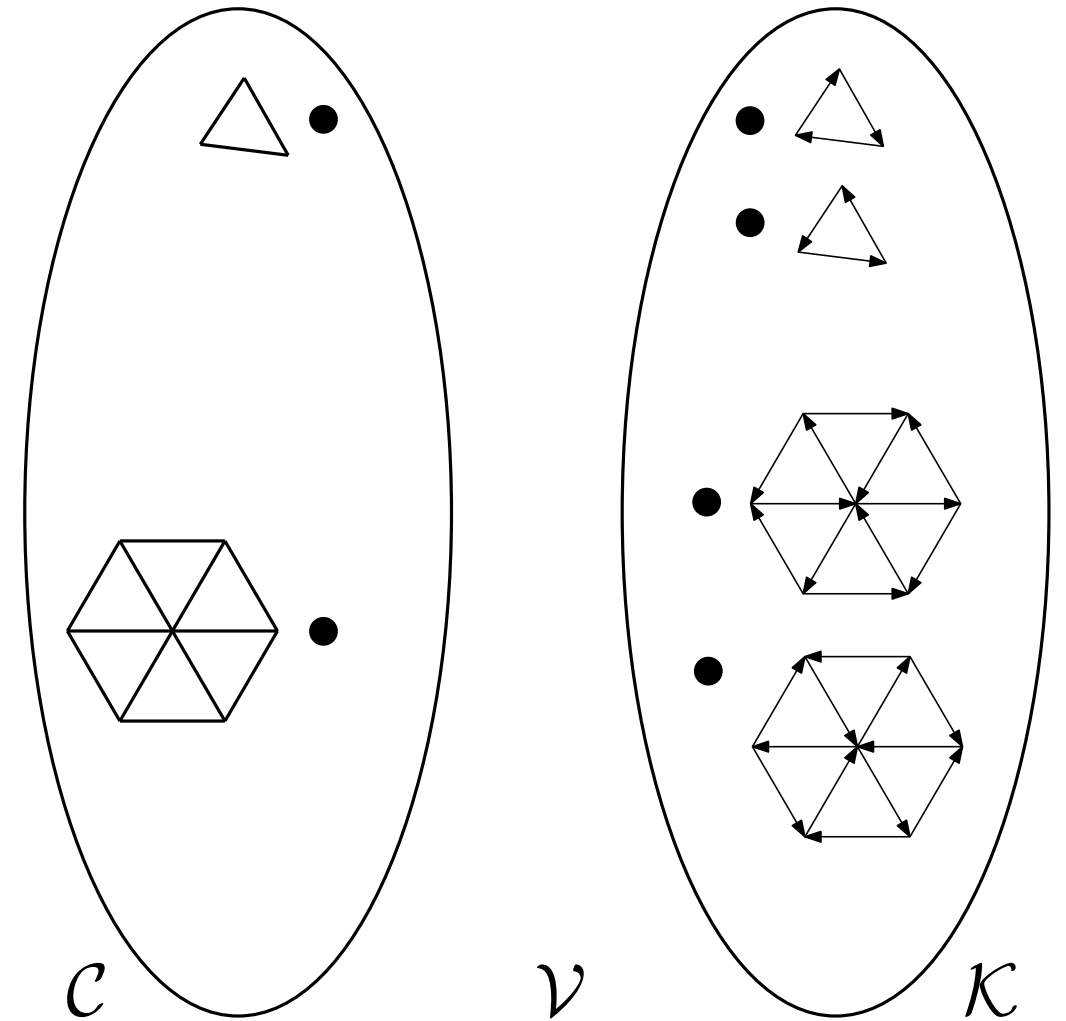
(Davis, Putnam, Robinson, Matiyasevich)
(Ladner)

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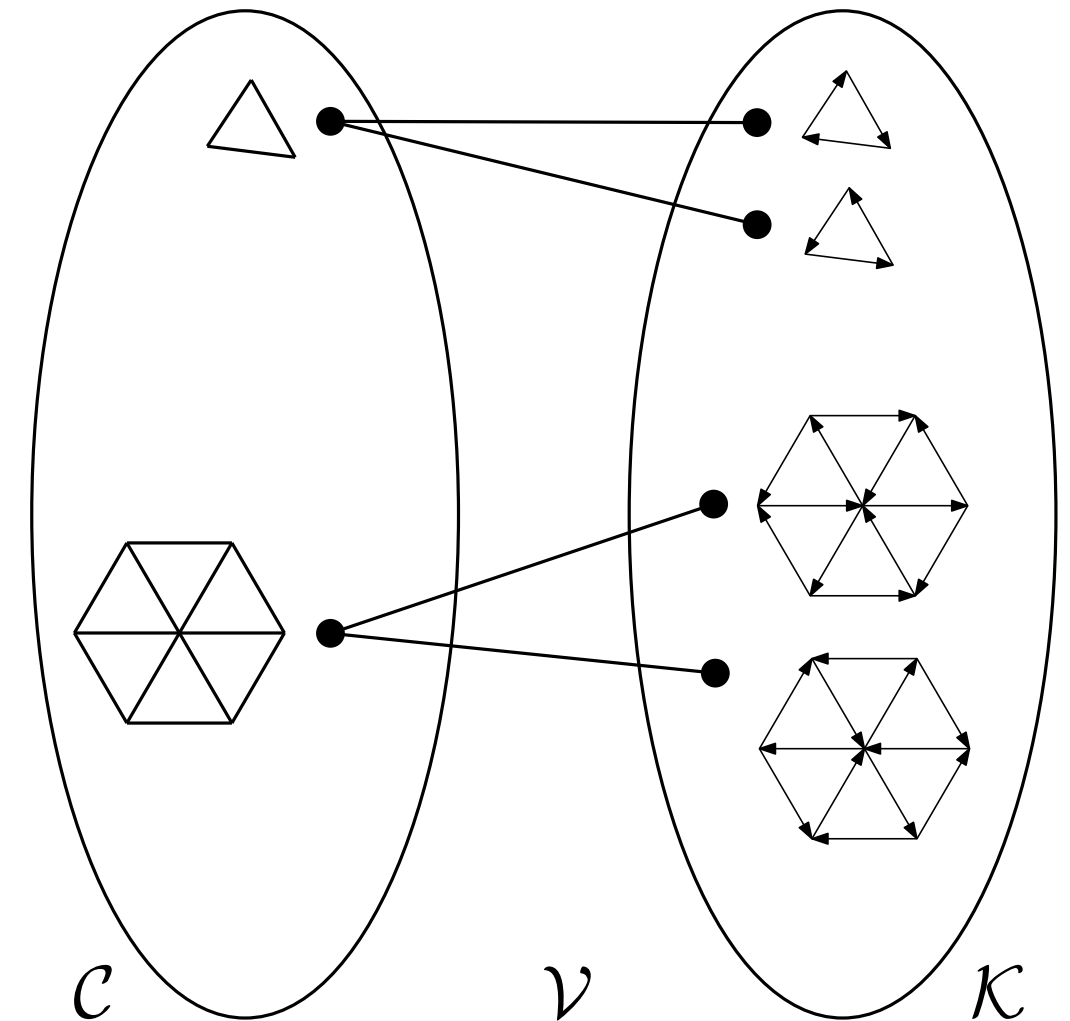


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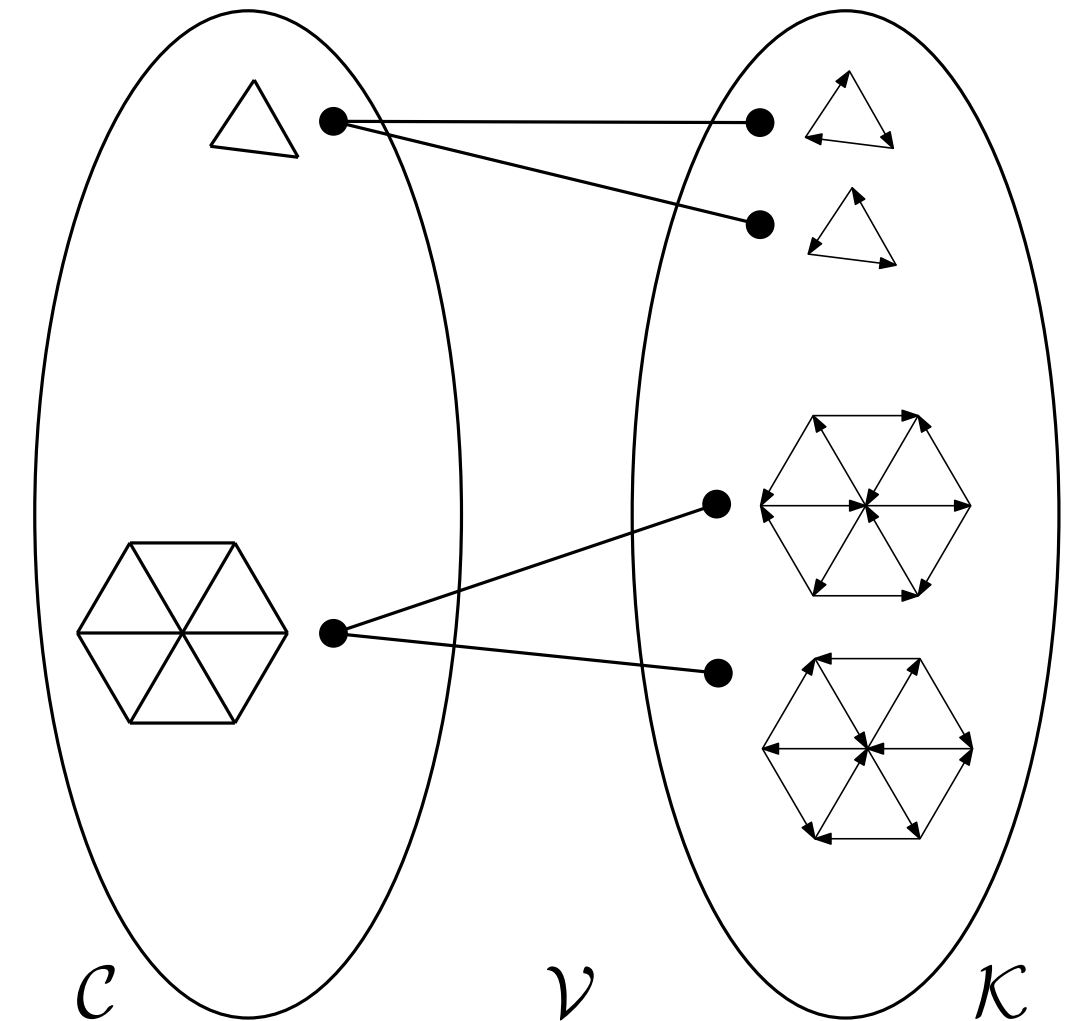
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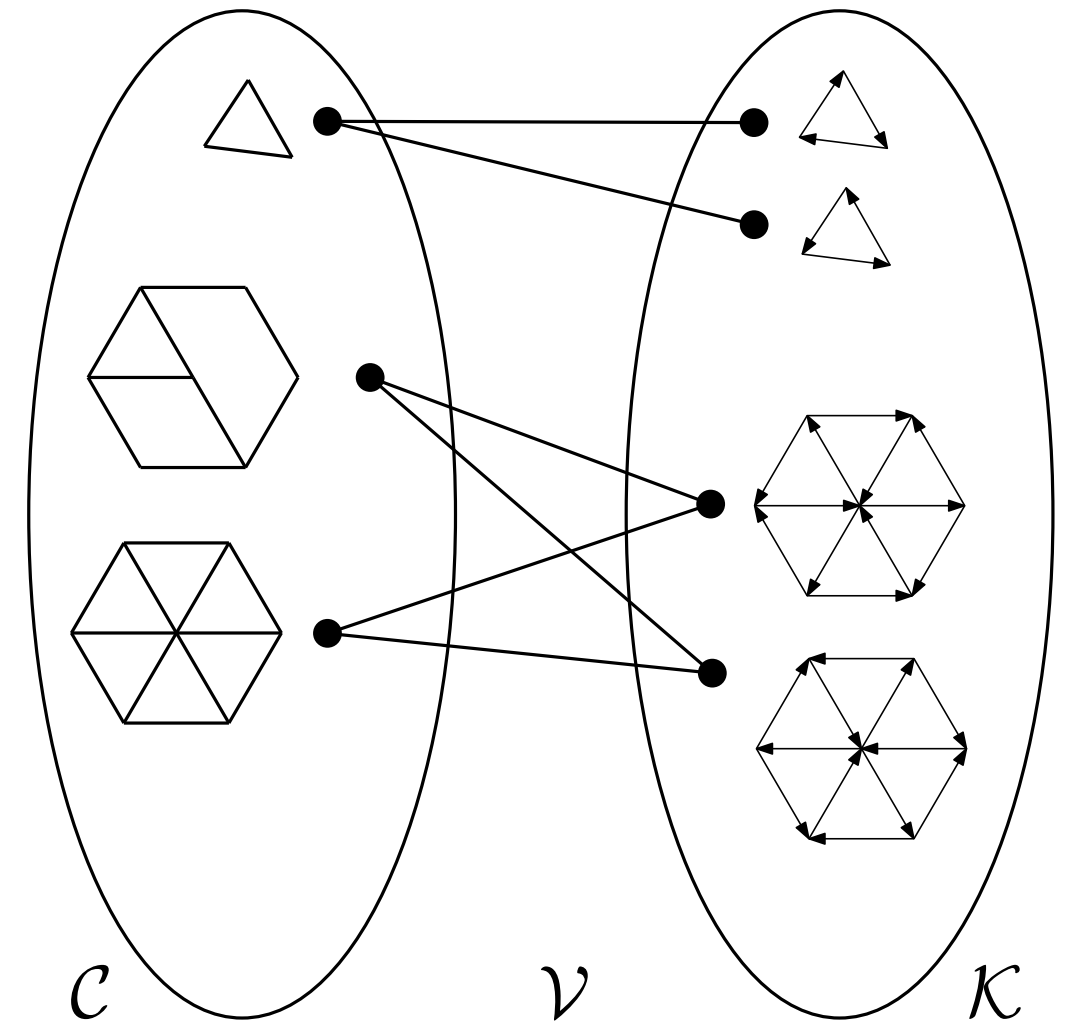


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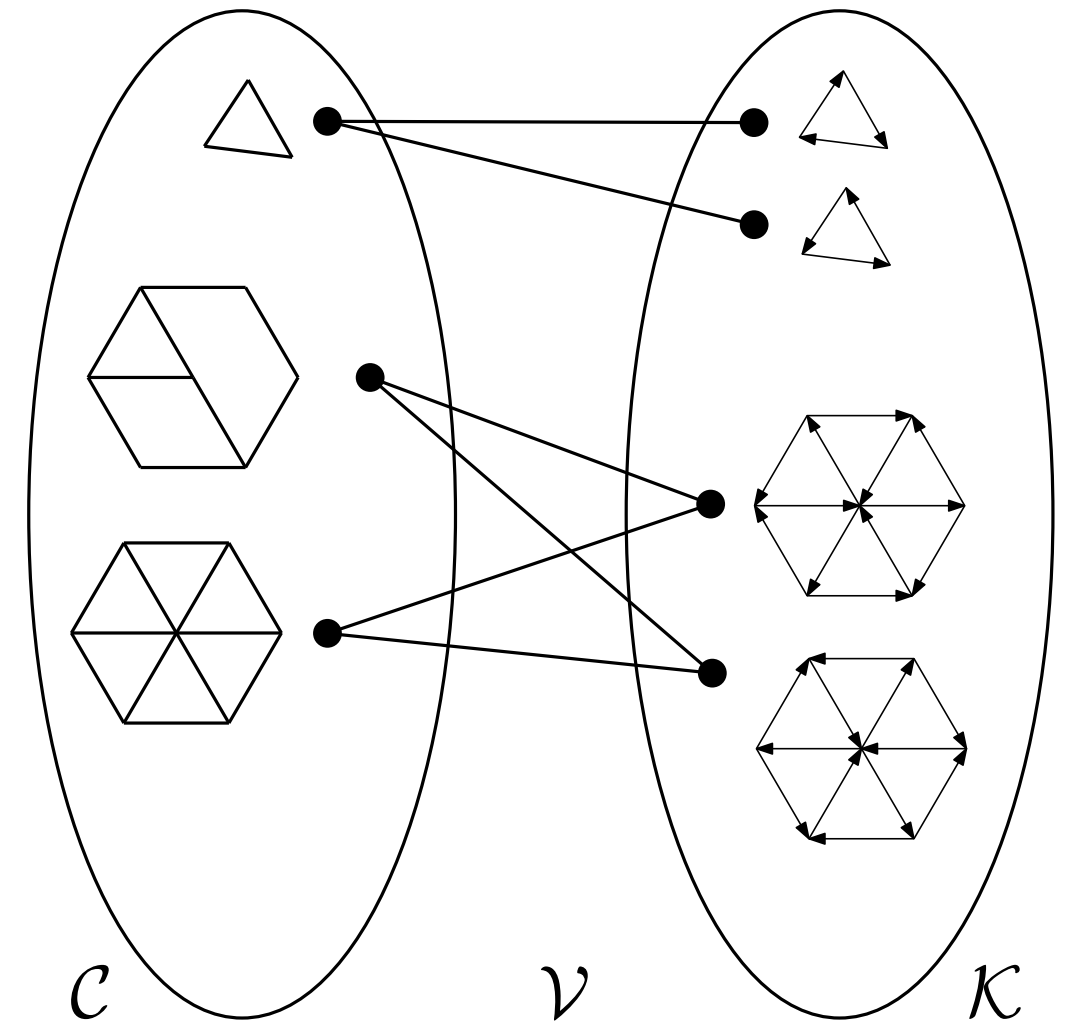


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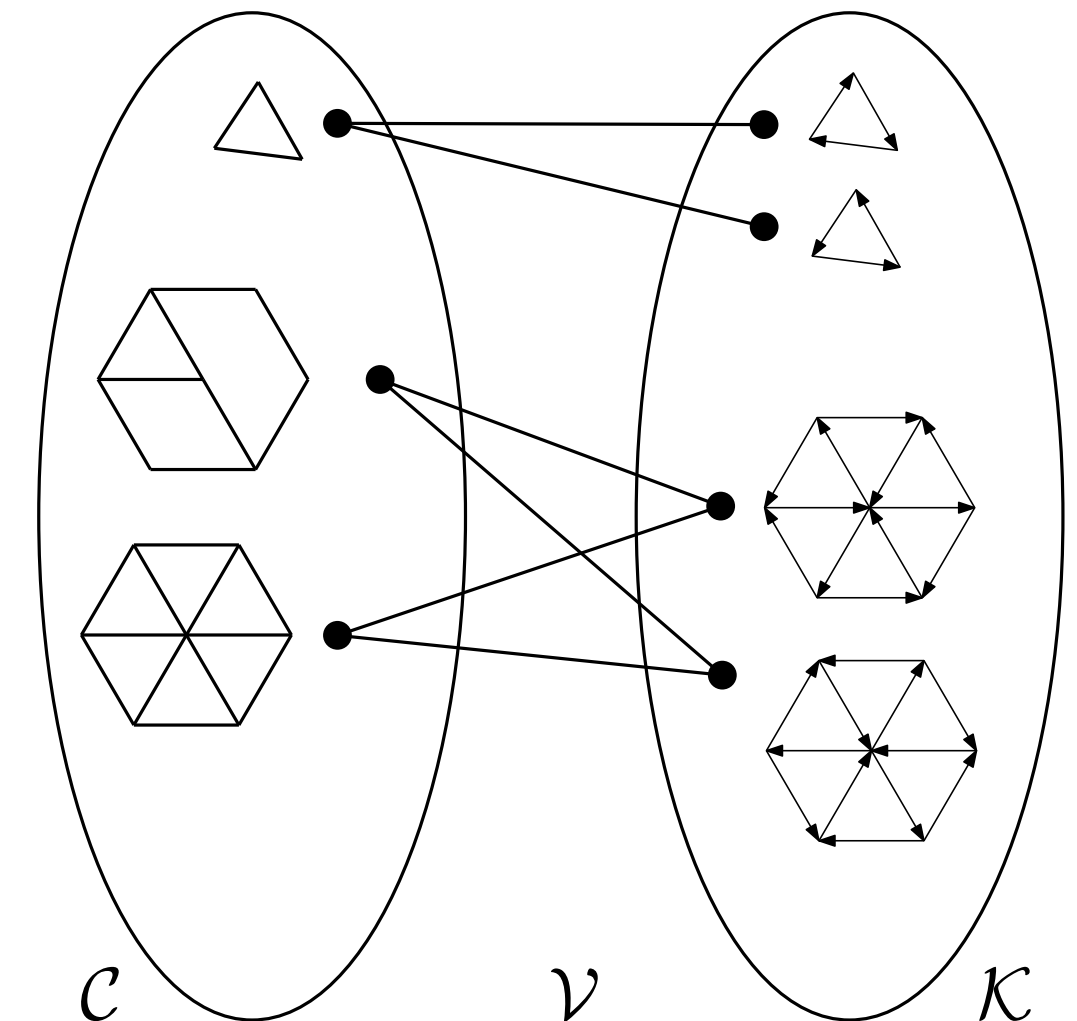
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Under those conditions:

- $\mathcal{C} = \text{CSP}(\mathbb{A})$, where \mathbb{A} is a **reduct of a finitely bounded homogeneous structure** \mathbb{A}^* .
- $\text{Aut}(\mathbb{A})$ is oligomorphic / \mathbb{A} is ω -categorical



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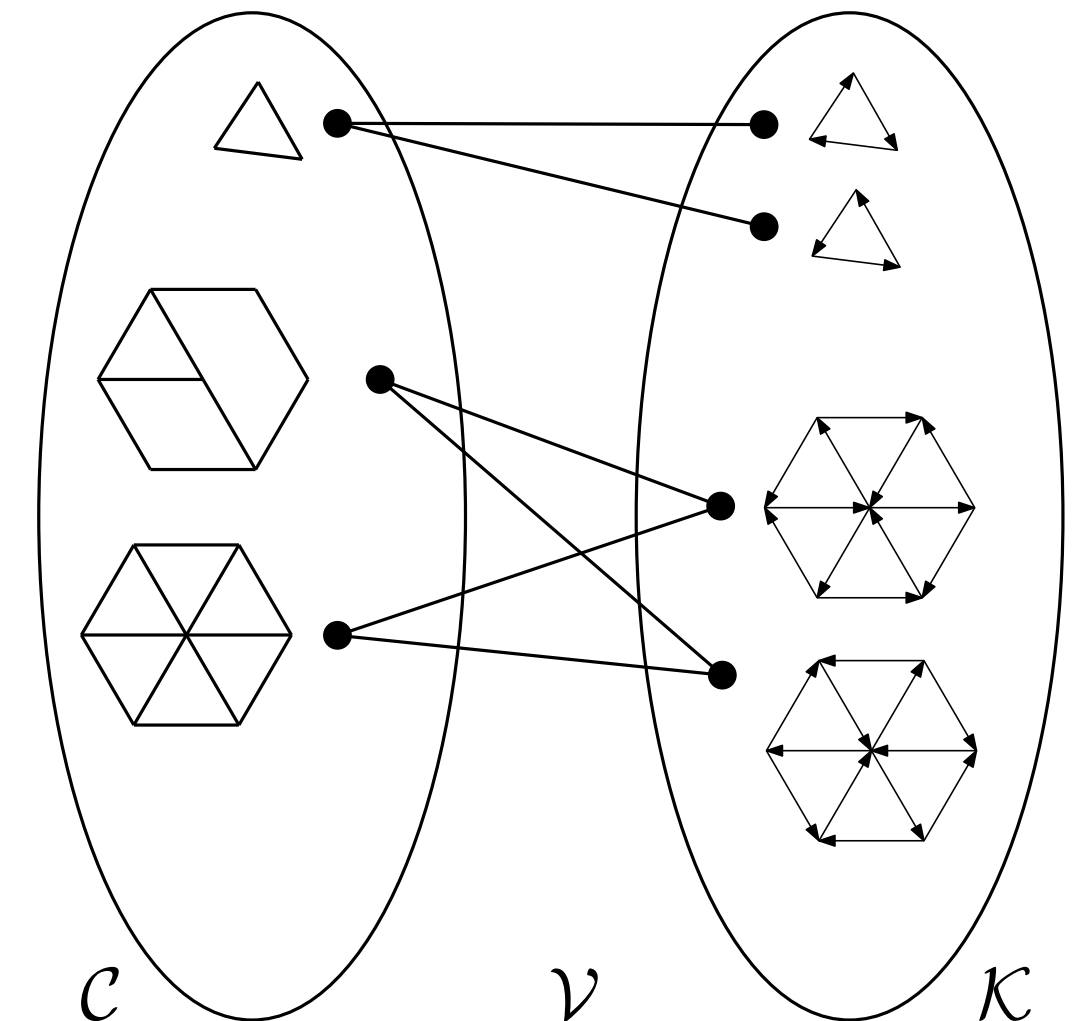
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Then \mathcal{C} is in P or NP-complete.

Non-examples:

- \mathcal{C} = satisfiable systems of affine equations over \mathbb{Z}
- ...feasible linear programs $Ax \leq b$
- ...systems of satisfiable polynomial equations over \mathbb{Q}



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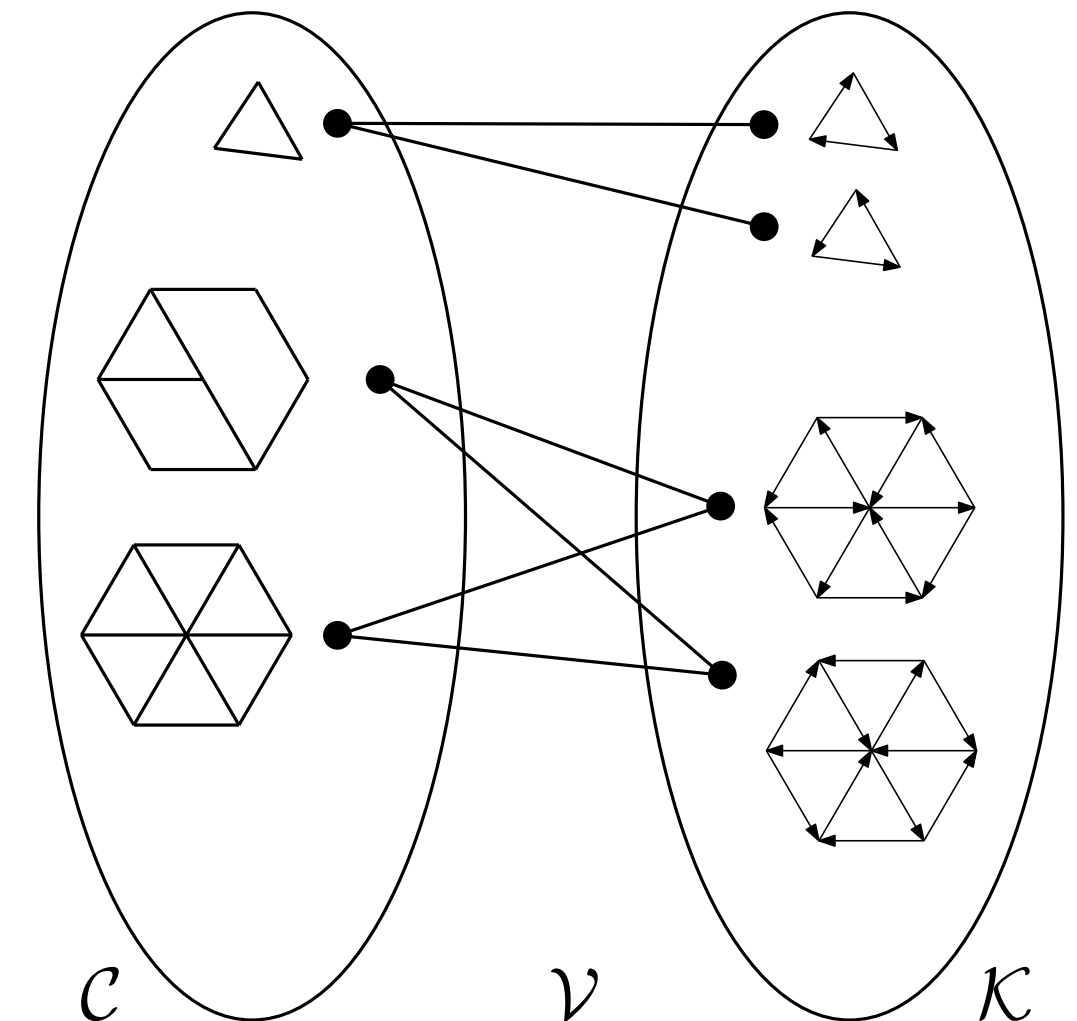
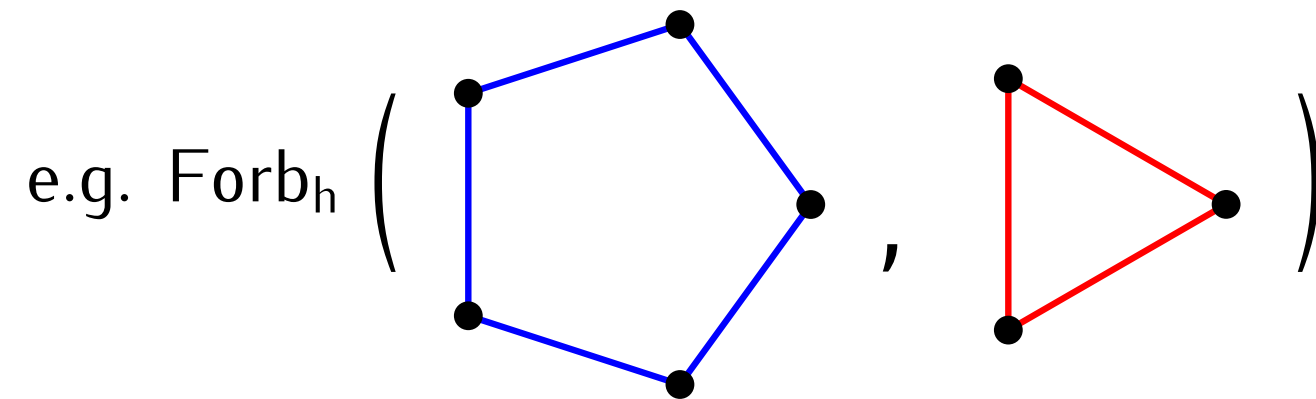
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Then \mathcal{C} is in P or NP-complete.

Examples:

- Order CSPs
- **GMSNP**: problems of coloring edges of a graph while avoiding forbidden configurations



(\subseteq means non-induced substructure)

Problem. Betweenness is the following problem:

Input: variables V and constraints $Betw(x, y, z)$

Question: is there a linear order on V such that $x < y < z$ or $z < y < x$ for every constraint?

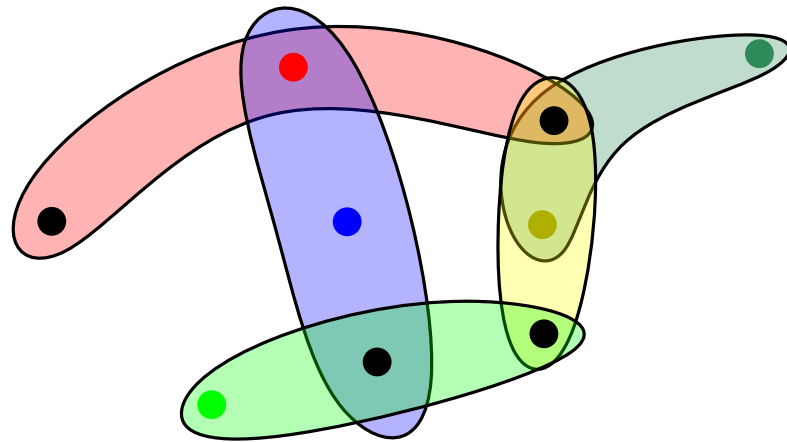
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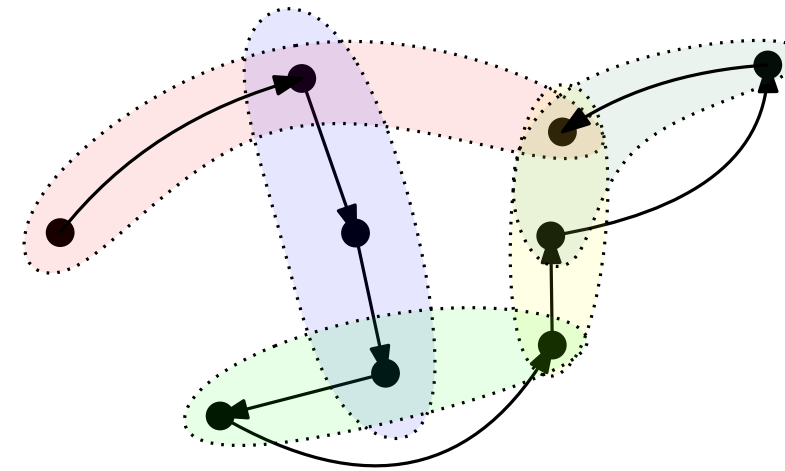
Question: is there a linear order on V such that $x < y < z$ or $z < y < x$ for every constraint?

The class \mathcal{C} of yes-instances satisfies the conditions of the conjecture:

Instance of the Betweenness



Satisfiability certificate



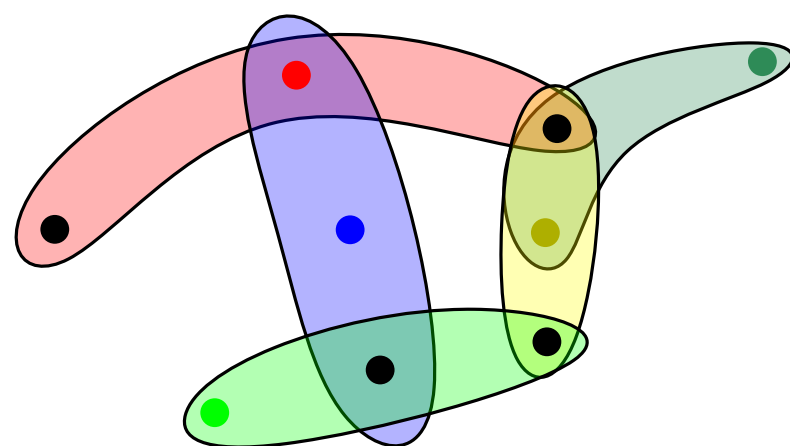
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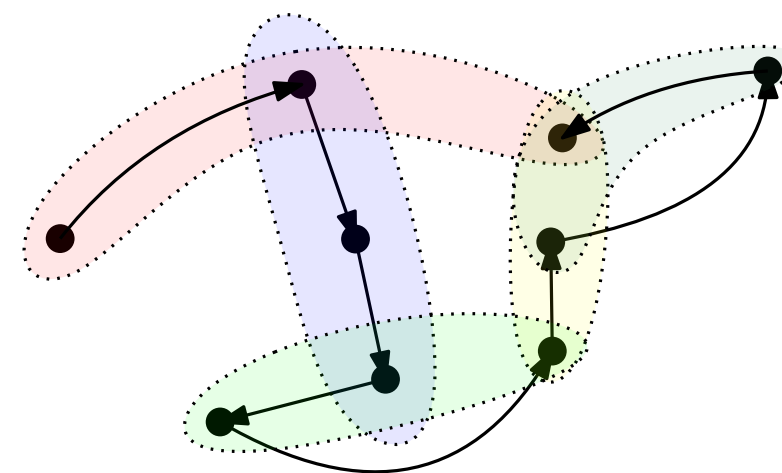
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By Fraïssé's theorem, Betweenness is $\text{CSP}(\mathbb{Q}; Betw)$

$$Betw = \{(a, b, c) \mid a < b < c \vee c < b < a\}$$

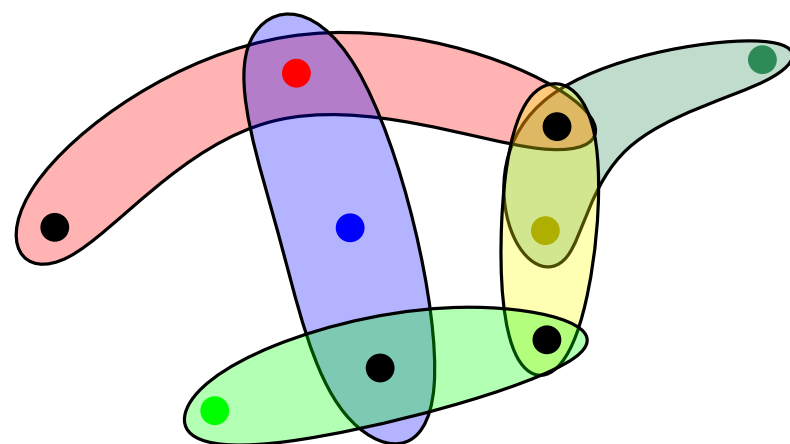
Problem. And/or precedence is the following problem:

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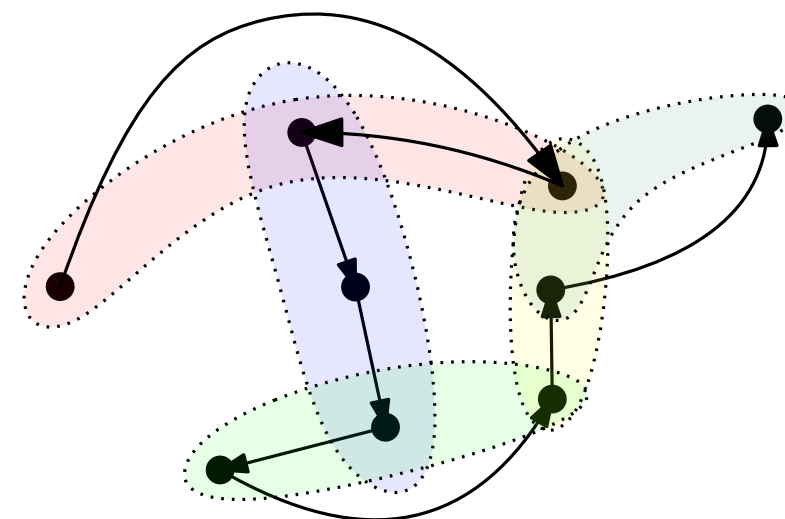
Question: is there a linear order on V such that $x > y \vee x > z$ for every constraint?

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Instance of and/or precedence



Satisfiability certificate



By Fraïssé's theorem, and/or precedence is $\text{CSP}(\mathbb{Q}; R)$

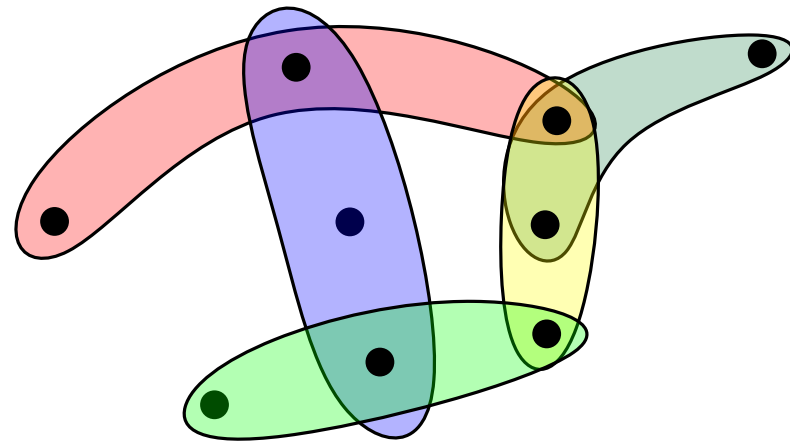
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The conjecture is proved when \mathcal{K} is...

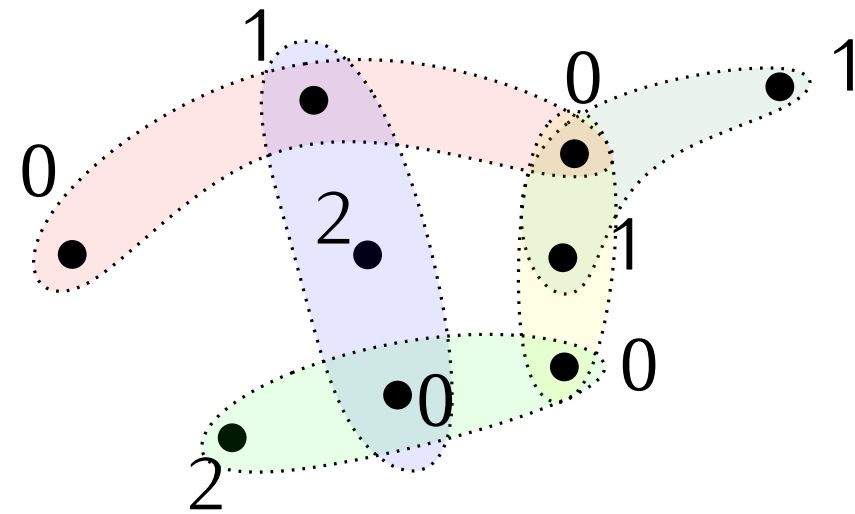
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(Bulatov) (Zhuk)

Instance of $\text{CSP}(\mathbb{A})$



Satisfiability certificate

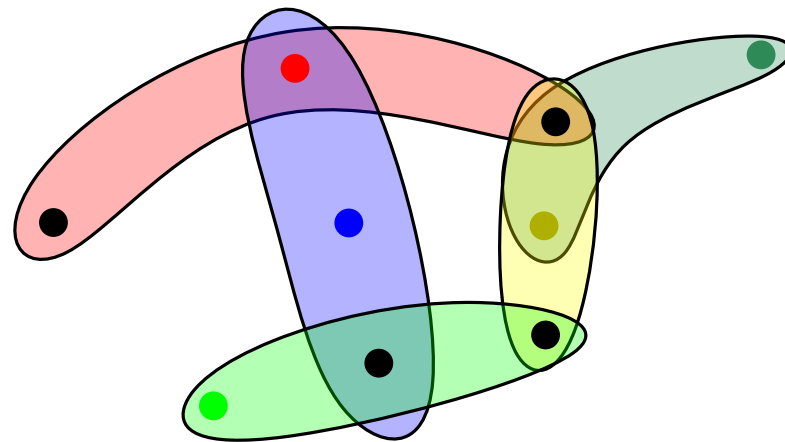


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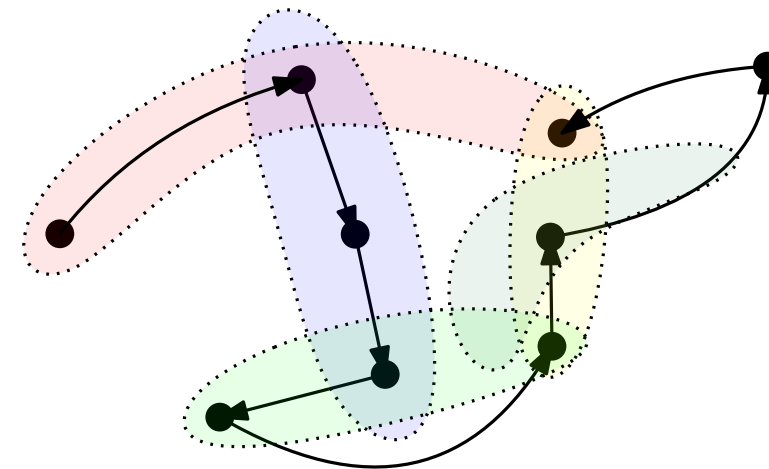
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Linear orders
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Satisfiability certificate



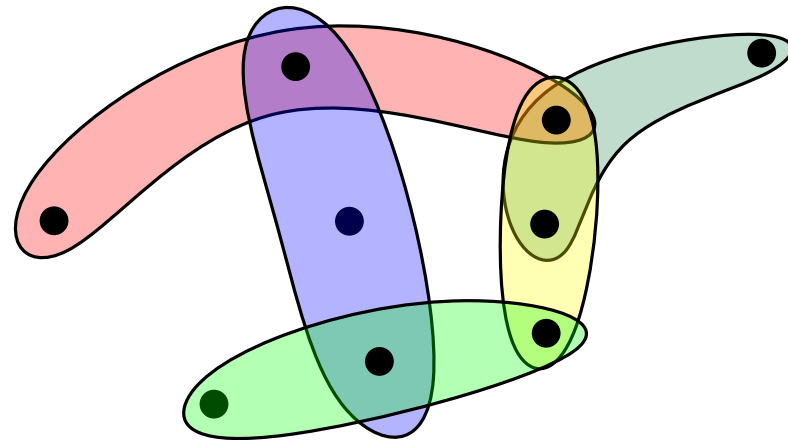
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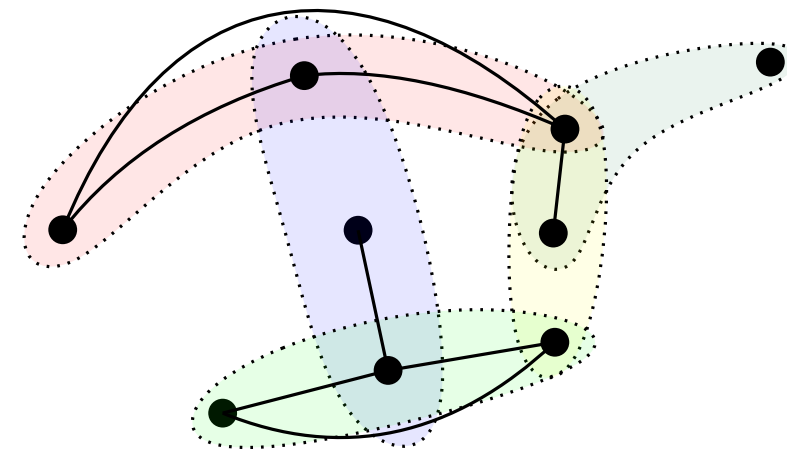
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Satisfiability certificate



The conjecture is proved when \mathcal{K} is...

Maps to a finite set
(Bulatov) (Zhuk)



Unary structures
(Bodirsky, M.)



Monadically stable
(Bodor)

Linear orders
(Bodirsky, Kára)



Partial orders
(Kompatscher, Pham)

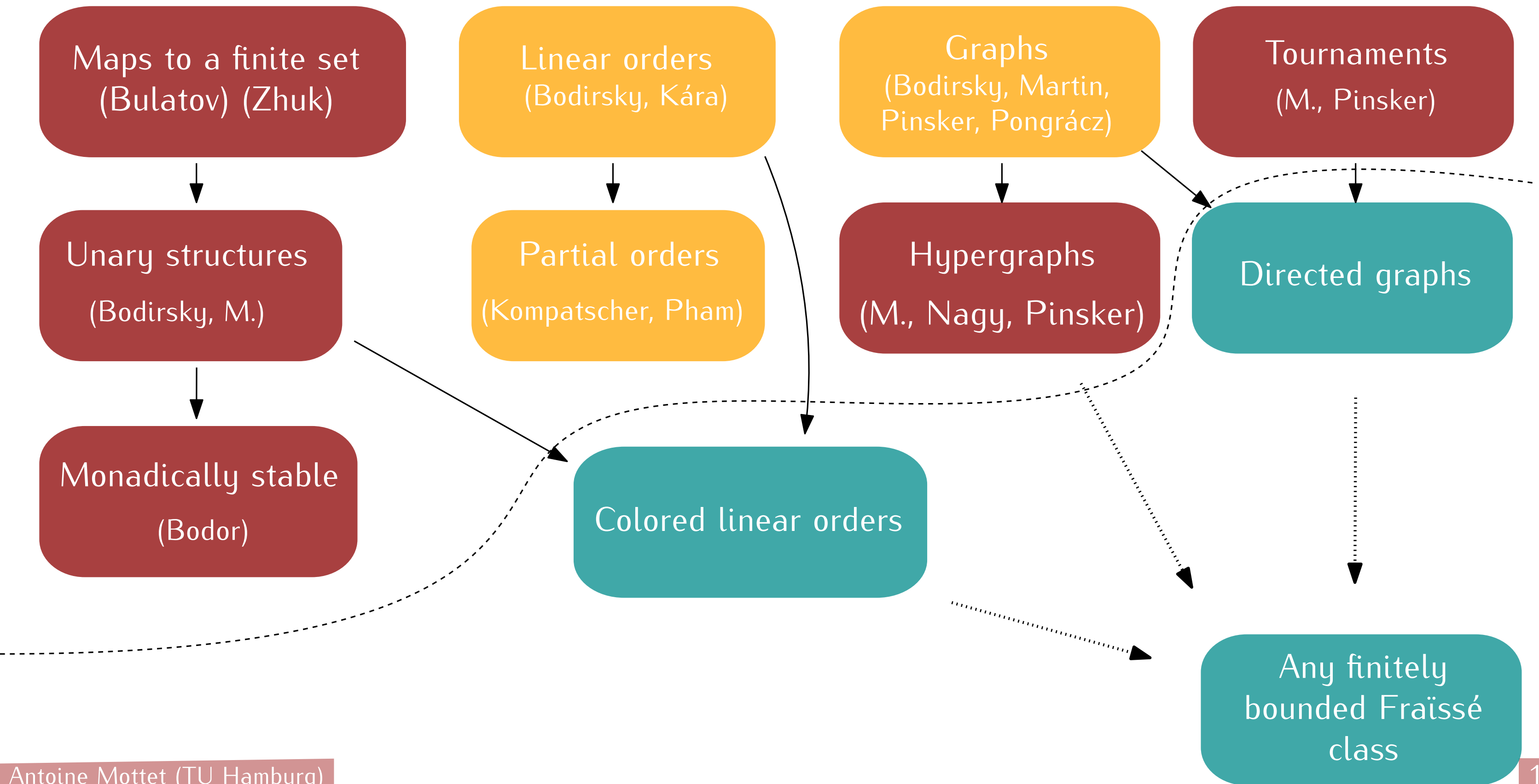
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Hypergraphs
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Tournaments
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The universal-algebraic approach: how to study CSPs

Definition. A **polymorphism** of \mathbb{A} is a function $f: \mathbb{A}^n \rightarrow \mathbb{A}$ such that for every relation $R^{\mathbb{A}}$ we have

$$\begin{array}{c} \mathbf{a}^1, \dots, \mathbf{a}^n \in R^{\mathbb{A}} \implies f(\mathbf{a}^1, \dots, \mathbf{a}^n) \in R^{\mathbb{A}} \\ \parallel \\ (f(a_1^1, \dots, a_1^n), \dots, f(a_r^1, \dots, a_r^n)) \end{array}$$

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Examples:

- Any $R \subseteq \{0, 1\}^2$, $f = \text{majority}$

0	1	0		0	0	1	0		0
0	0	1		0	1	0	1		1
\cap	\cap	\cap			\cap	\cap	\cap		
R	R	R			R	R	R		

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Examples:

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- Affine subspace $R \subseteq \{0, 1\}^n$, $f(x, y, z) = x + y + z$

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0	0	1		0	1	0	1		1
\cap	\cap	\cap			\cap	\cap	\cap		
R	R	R			R	R	R		

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R	R	R		

$\sum x_i = 5 \bmod 2$

Non-example:
 $R = \{(1, 0), (0, 1)\}$
 $f = \vee$

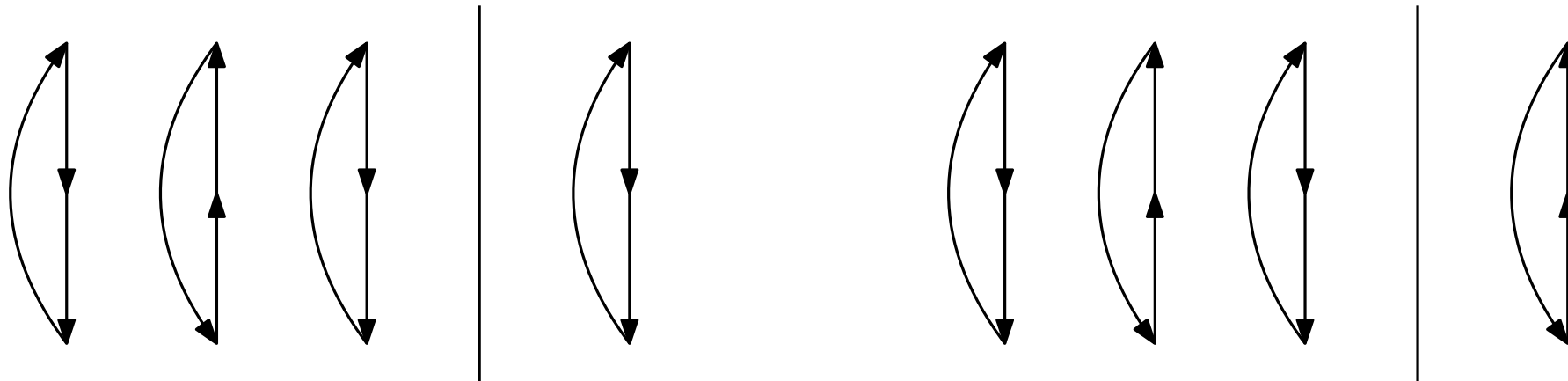
1	0		1
0	1		1
\cap	\cap		\nexists
R	R		R

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Examples:

- Cyclic orientations: both majority and minority



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Motto:

- $\text{Pol}(\mathbb{A})$ "rich" \Rightarrow can use polymorphisms to narrow down search space

majority

minority

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = f(x, x, x) \text{ or } f(x, x, y) = f(x, y, x) = f(y, x, x) = f(y, y, y)$$

- $\text{Pol}(\mathbb{A})$ "poor" \Rightarrow instances of $\text{CSP}(\mathbb{A})$ can have complicated solution spaces

Theorem (Bodirsky, Nešetřil). Let \mathbb{A} be ω -categorical and $R \subseteq A^r$. Then R is pp-definable in \mathbb{A} iff R is preserved by $\text{Pol}(\mathbb{A})$.

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$$\begin{array}{cccc} & \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^n \\ & \parallel & \parallel & \parallel \\ \mathbf{b}^1 = & (a_1^1 & a_1^2 & \cdots & a_1^n) \\ & & & & \\ & \vdots & \vdots & & \vdots \\ \mathbf{b}^r = & (a_r^1 & a_r^2 & \cdots & a_r^n) \end{array}$$

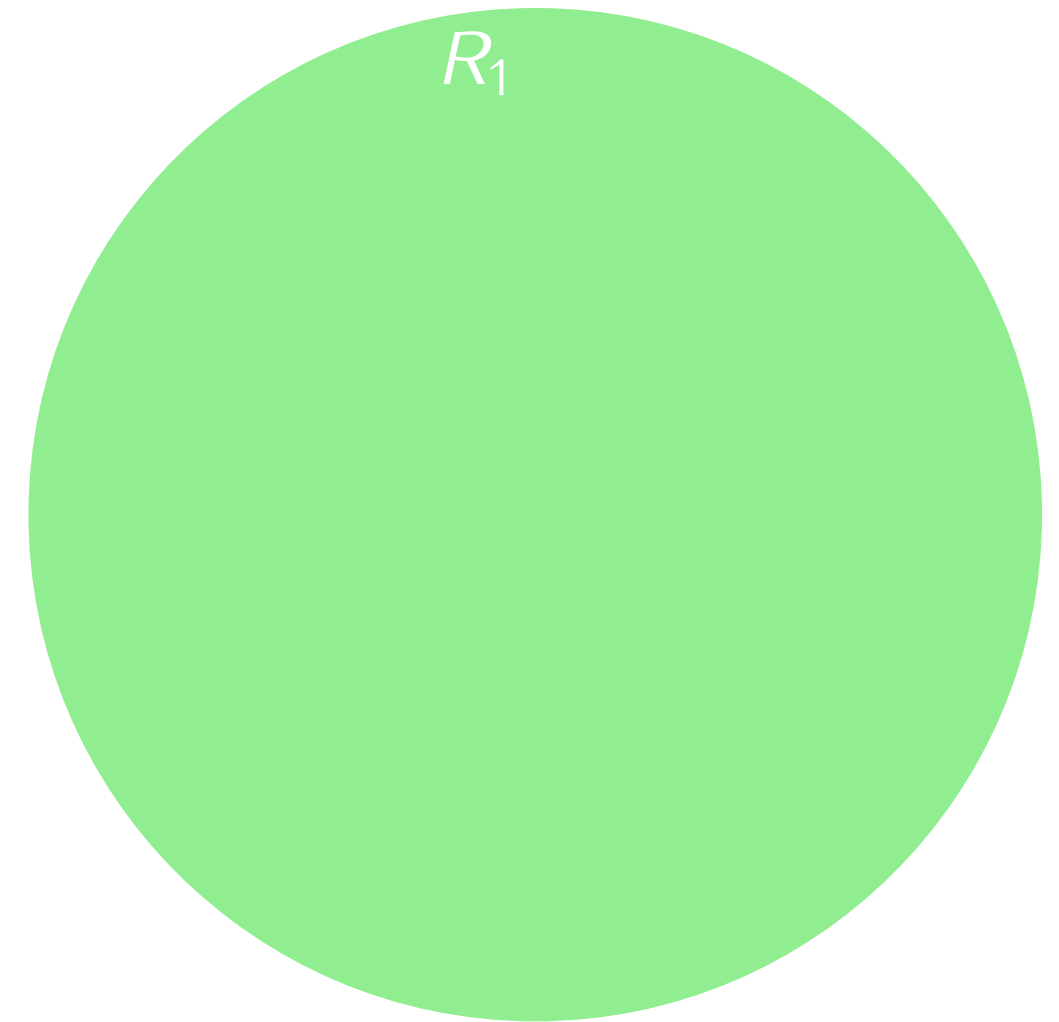
Lemma. Let \mathbb{X} be a finite structure, $x_1, \dots, x_n \in X$.

There exists a pp-formula $\varphi_{\mathbb{X}}(x_1, \dots, x_n)$ such that for all \mathbb{A} :

$$\mathbb{A} \models \varphi_{\mathbb{X}}(a_1, \dots, a_n) \iff \exists h: \mathbb{X} \rightarrow \mathbb{A} \forall i: h(x_i) = a_i$$

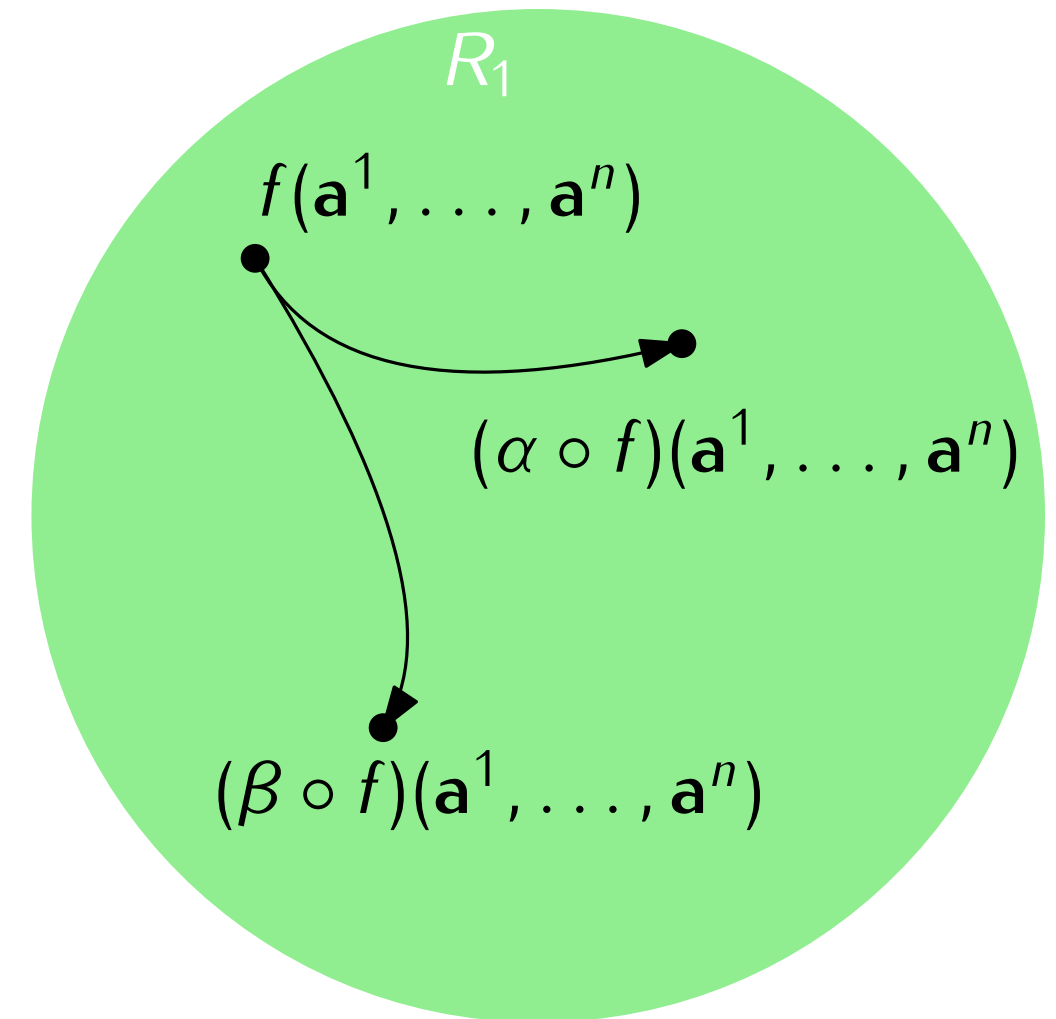
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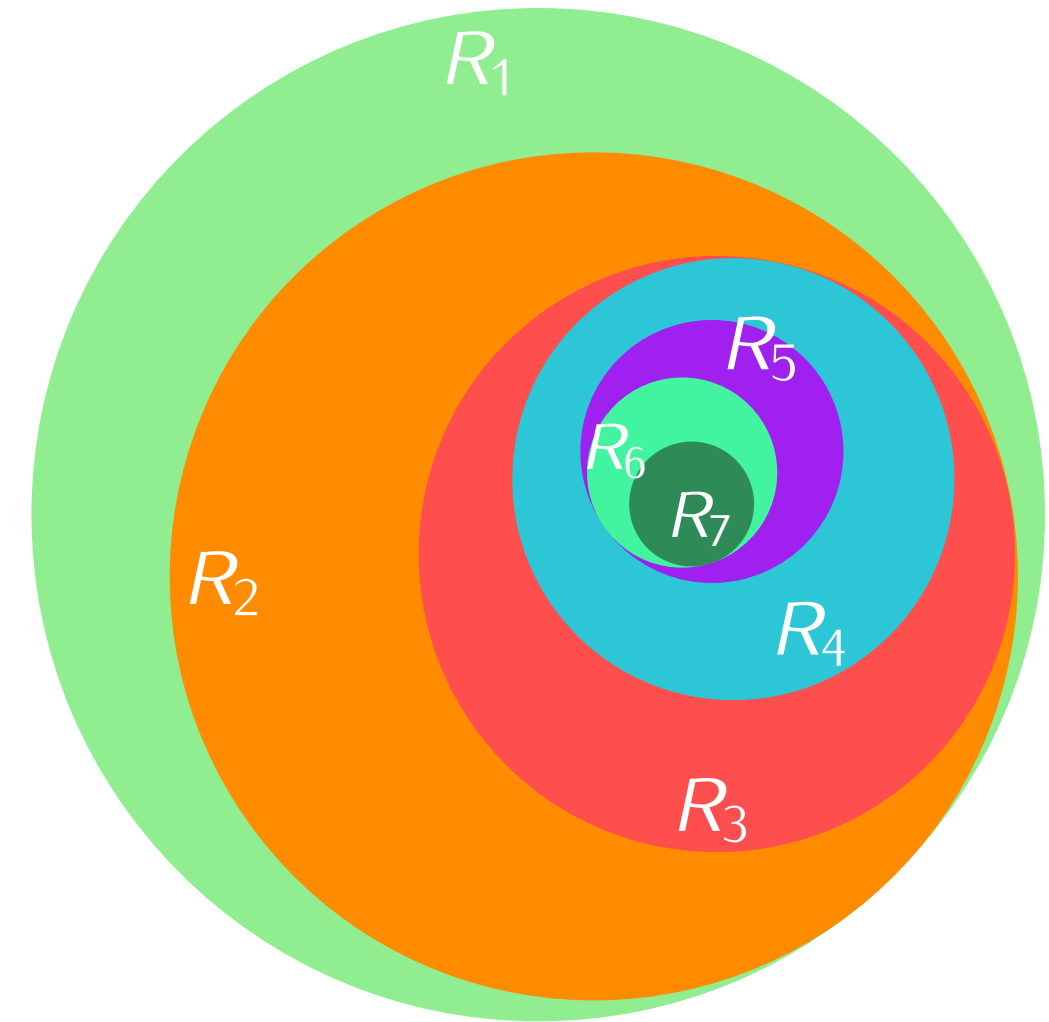
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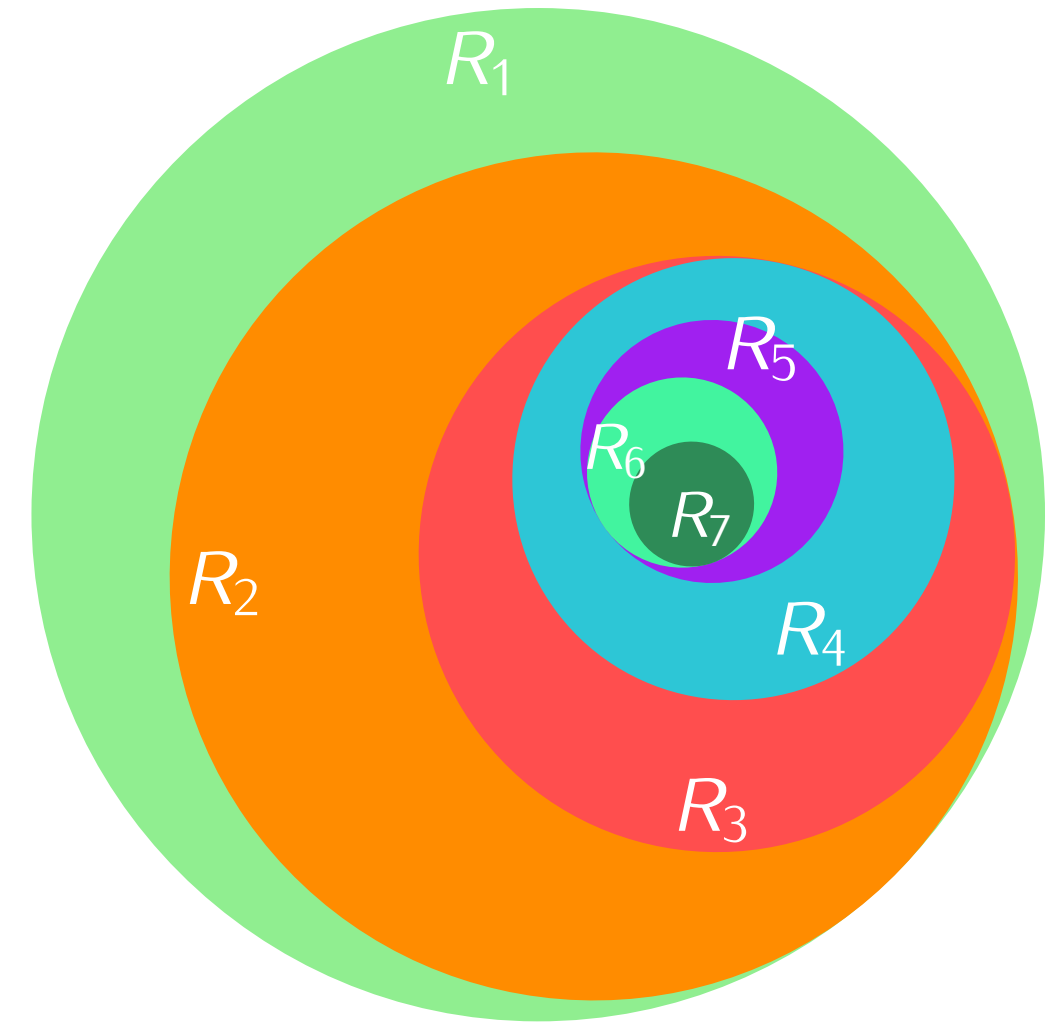
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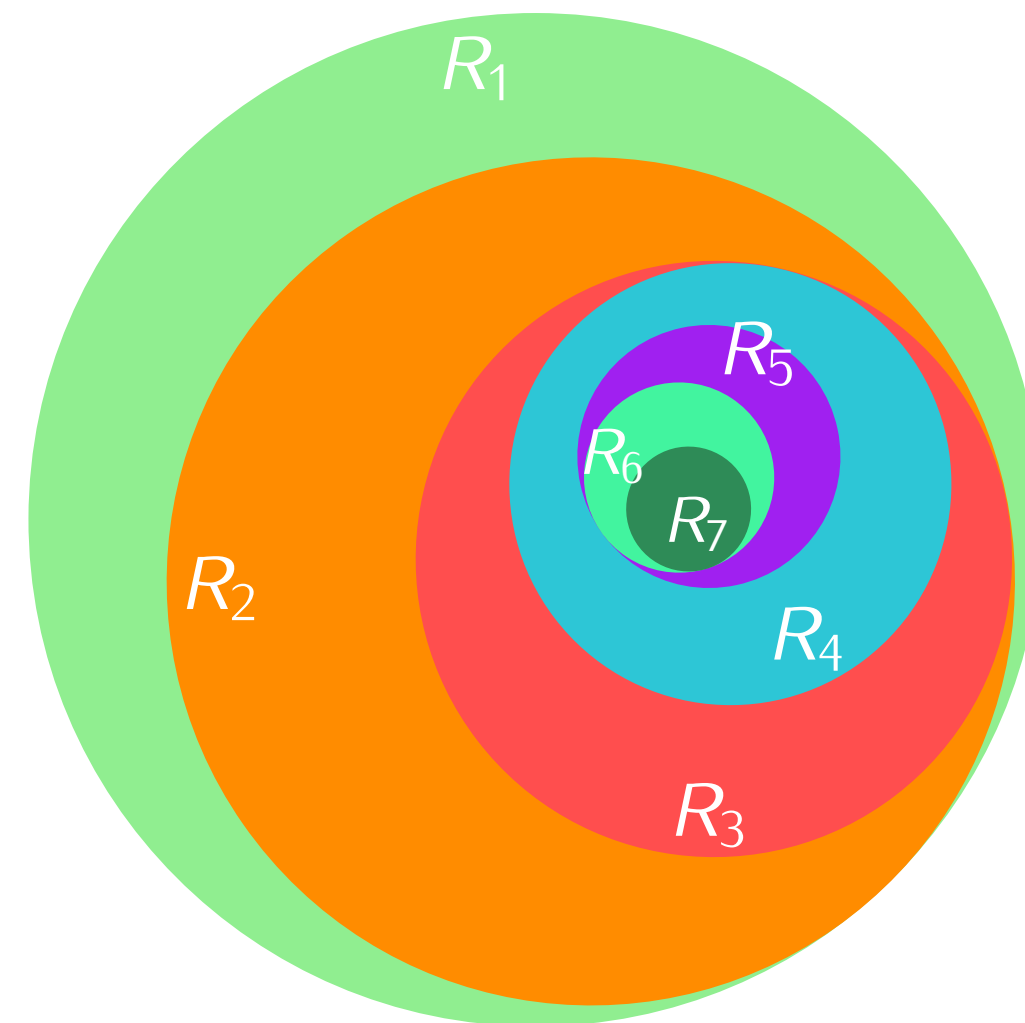
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This is the **standard compactness argument**.



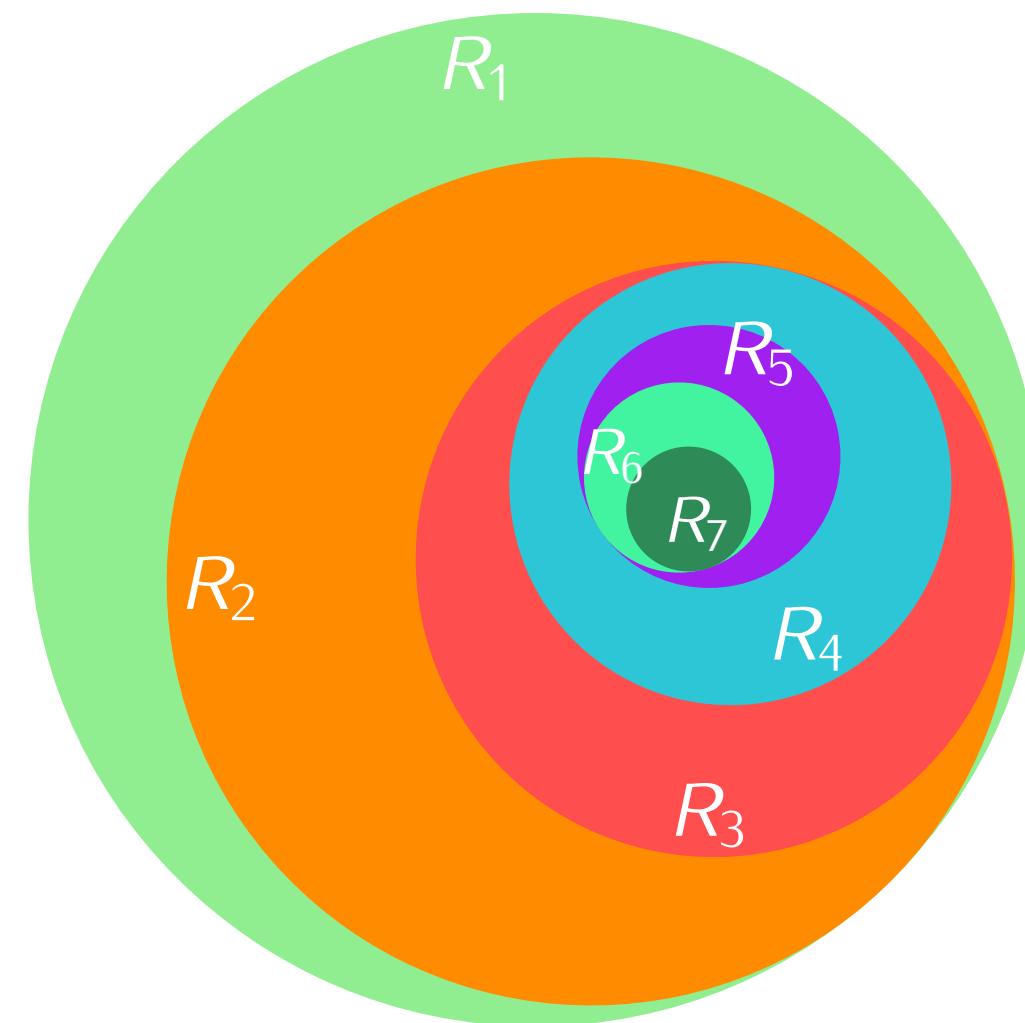
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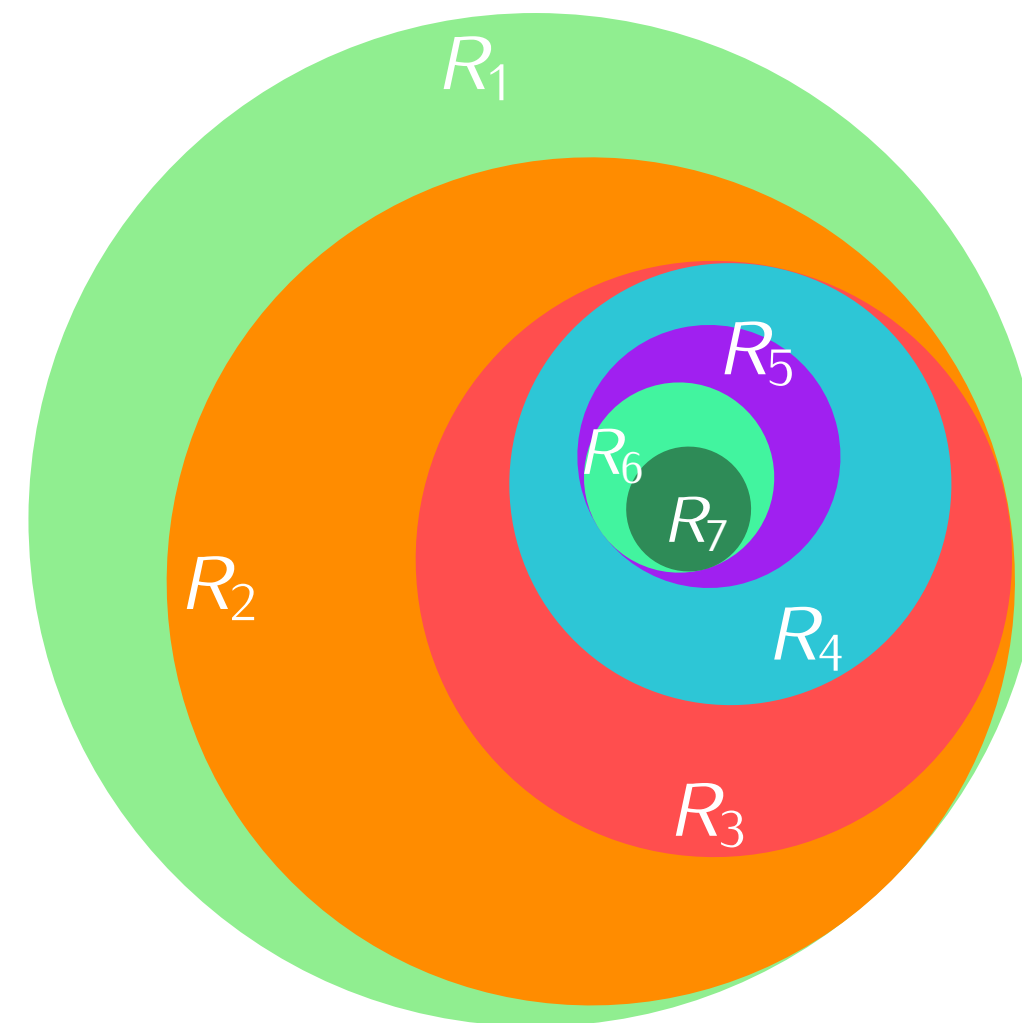
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- We can always find finitely many generators, by ω -categoricity



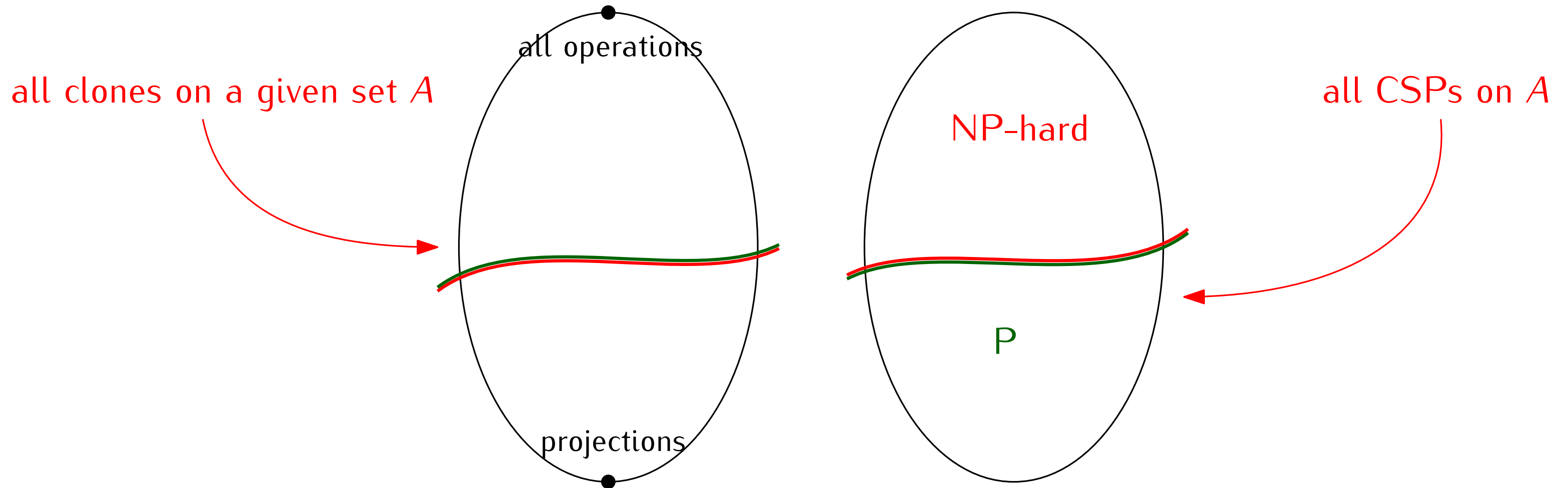
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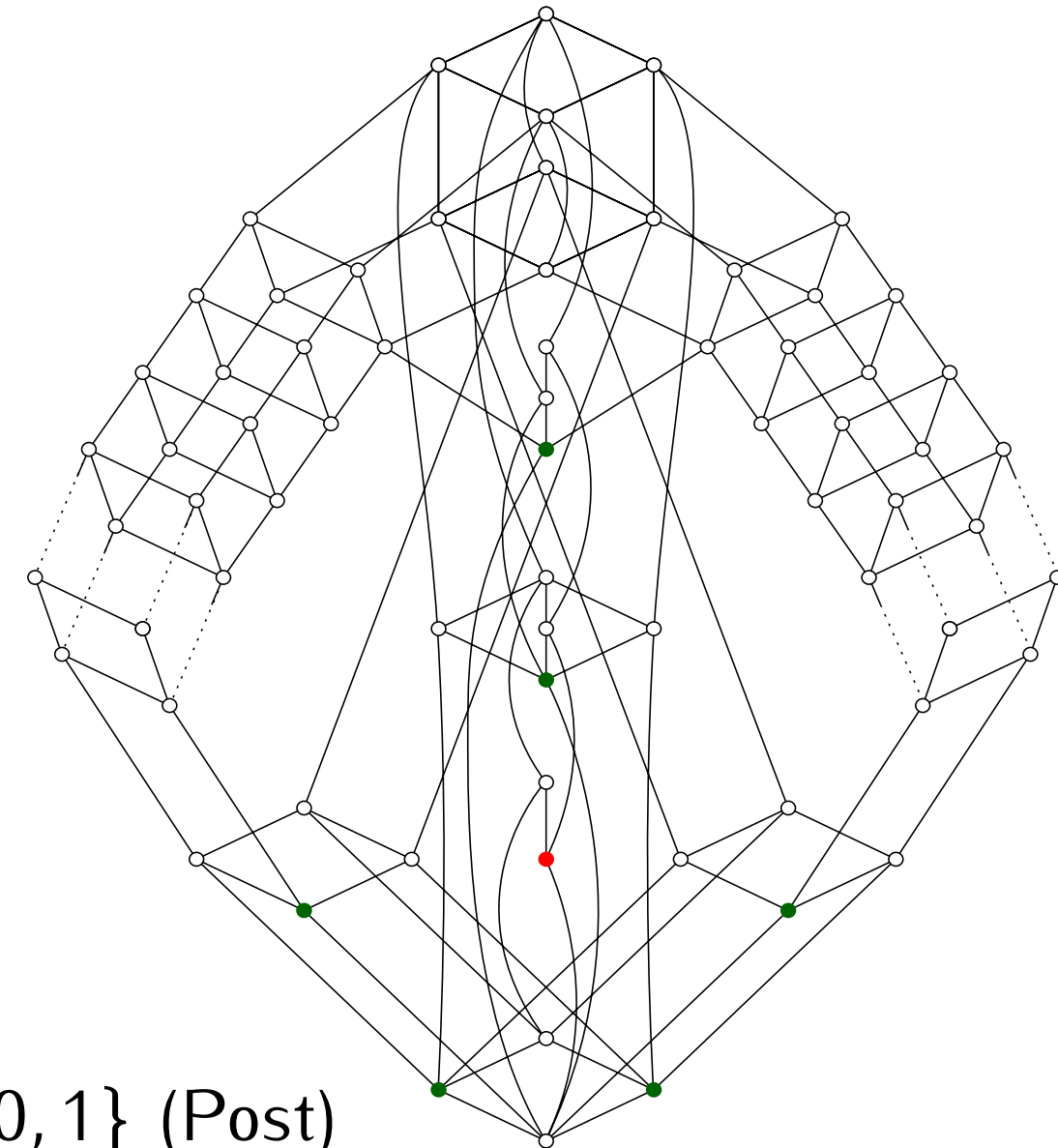


This order is way too fine, no hope of understanding it all (already continuum many clones on $\{0, 1, 2\}$)

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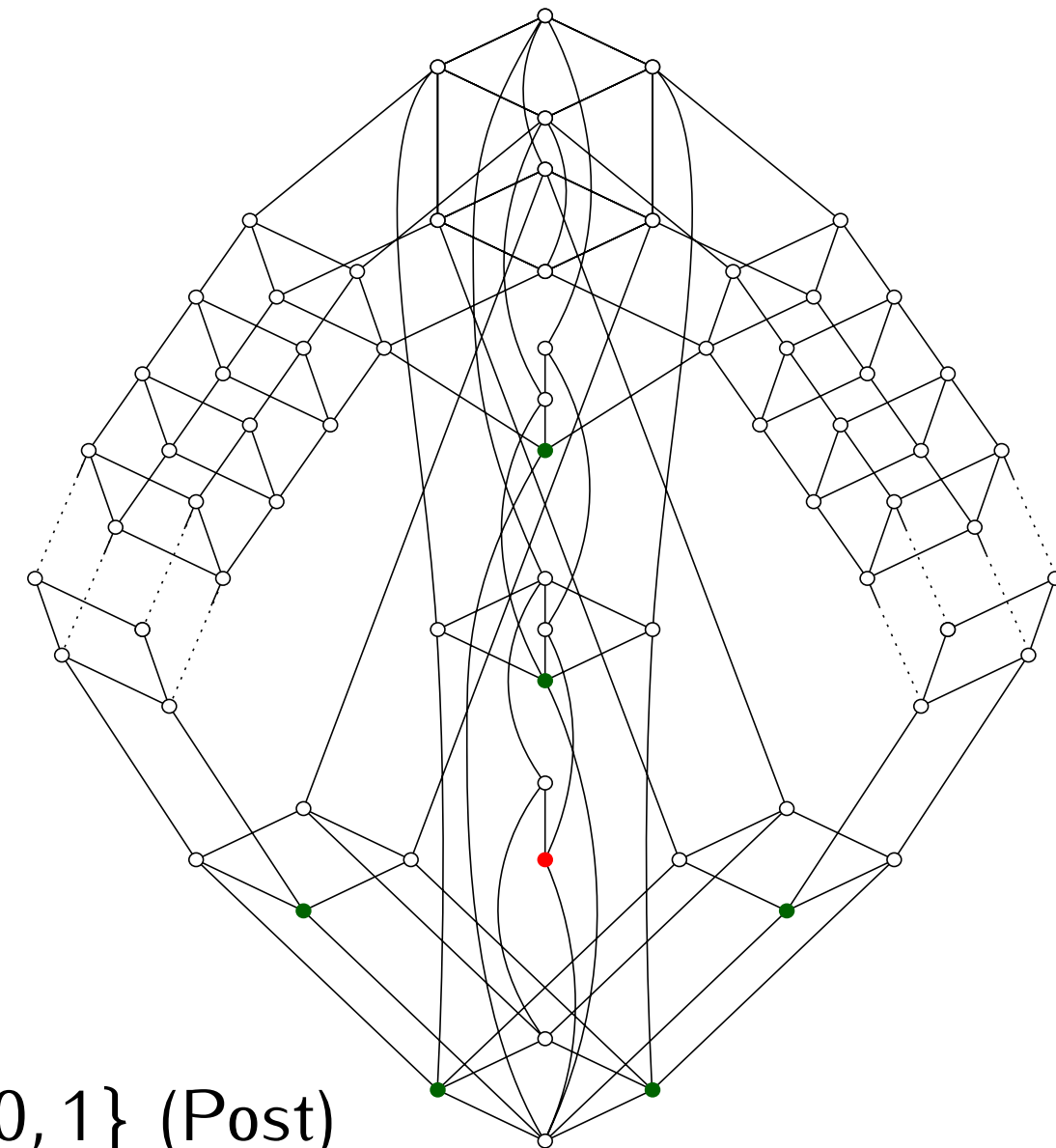
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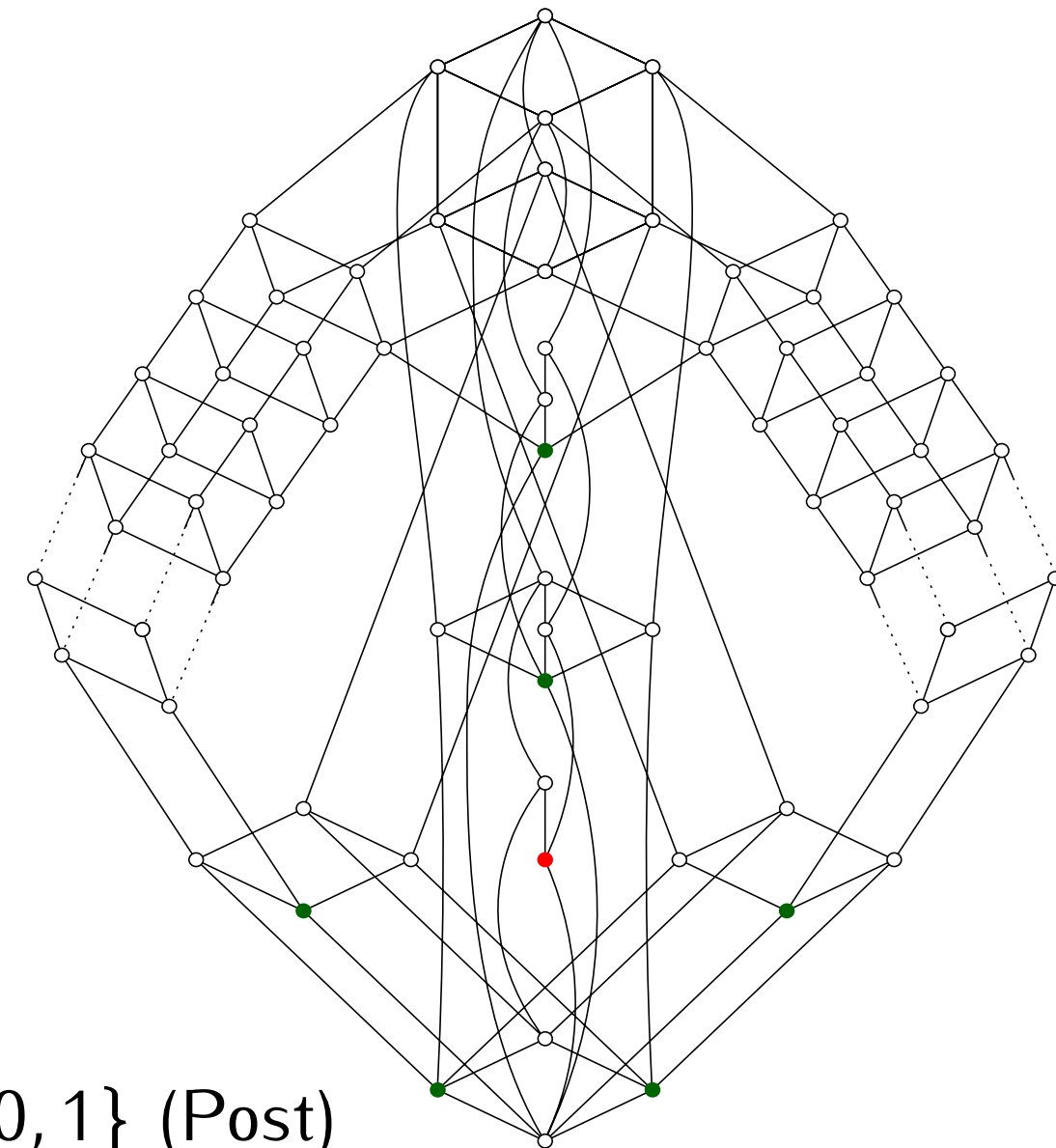
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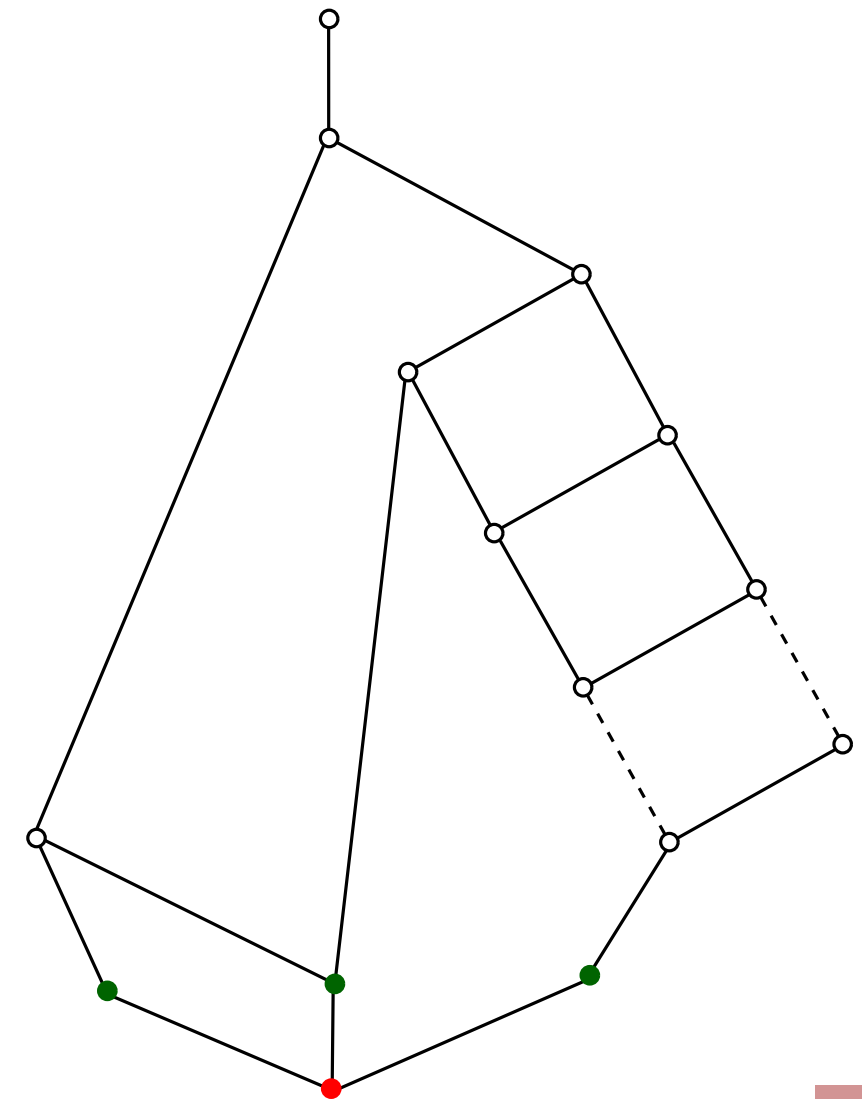
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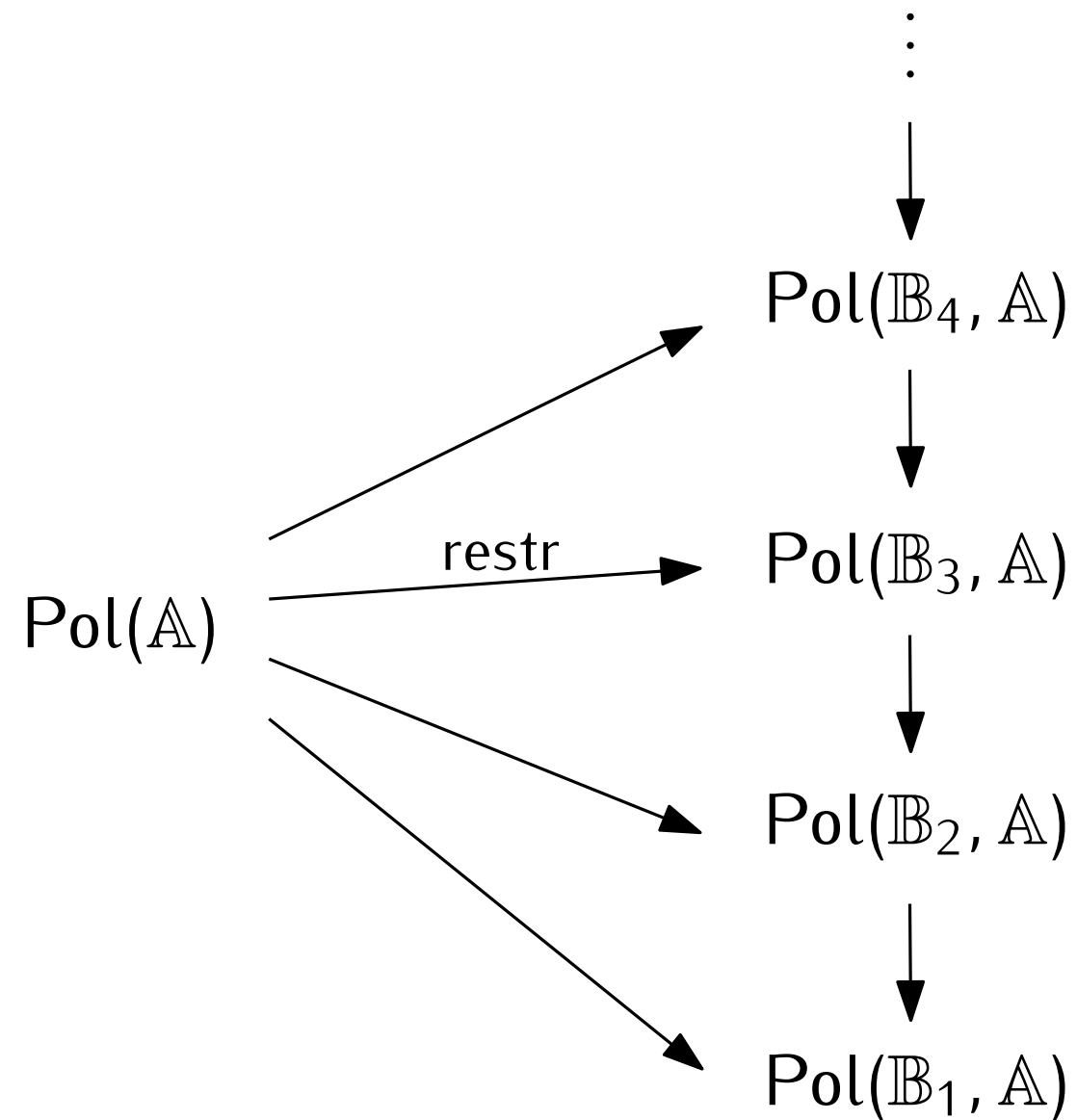
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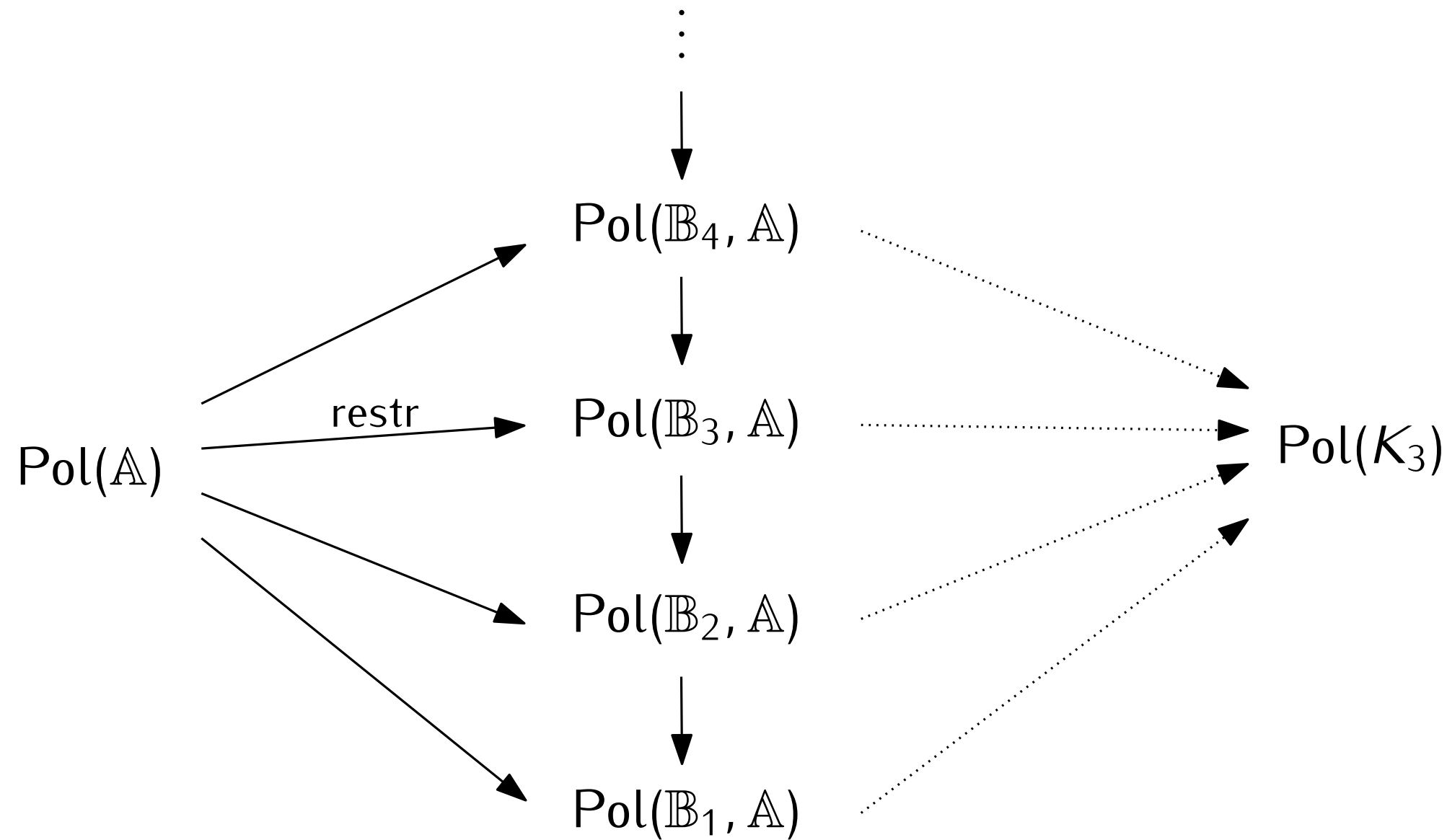
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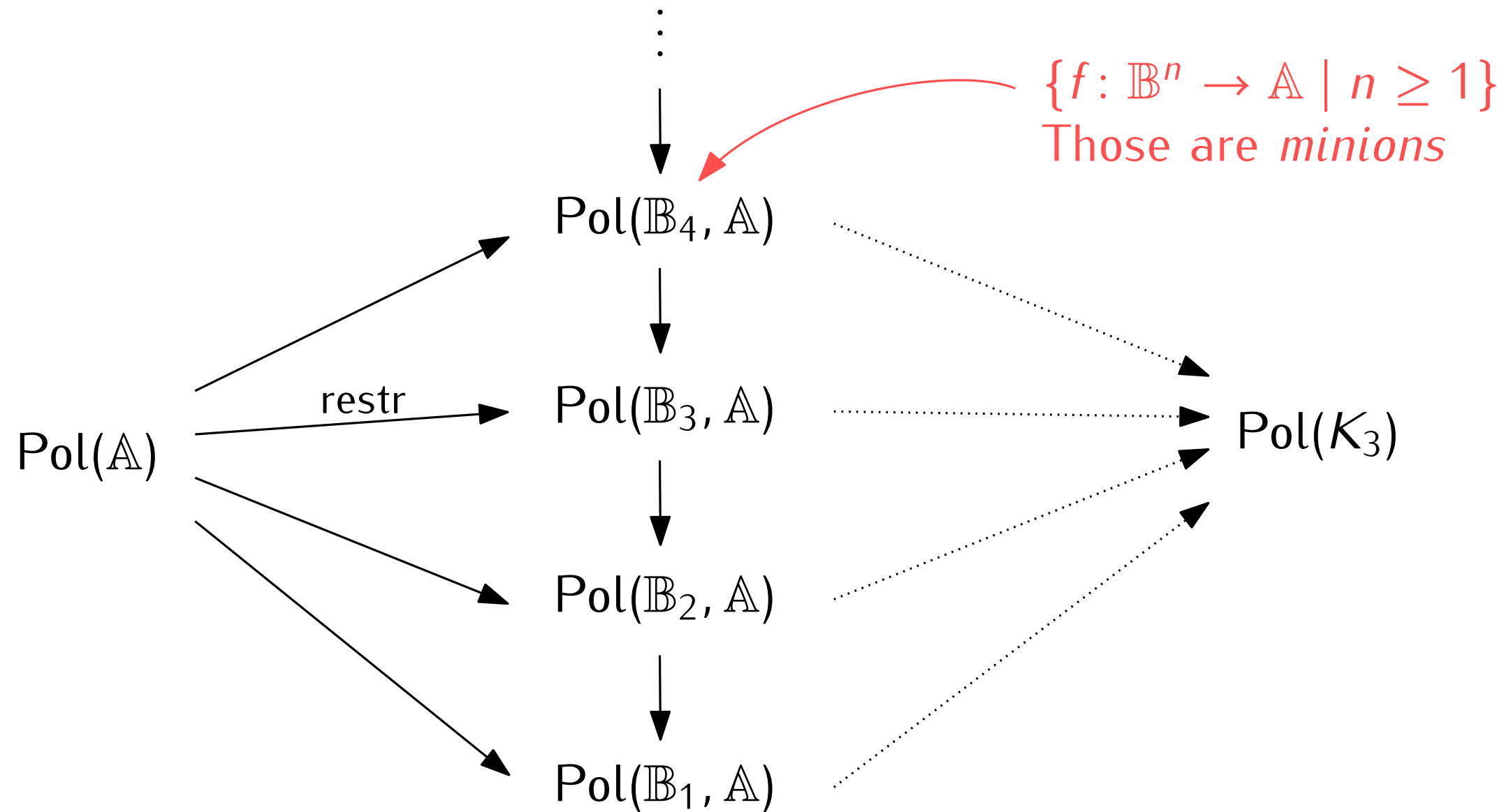
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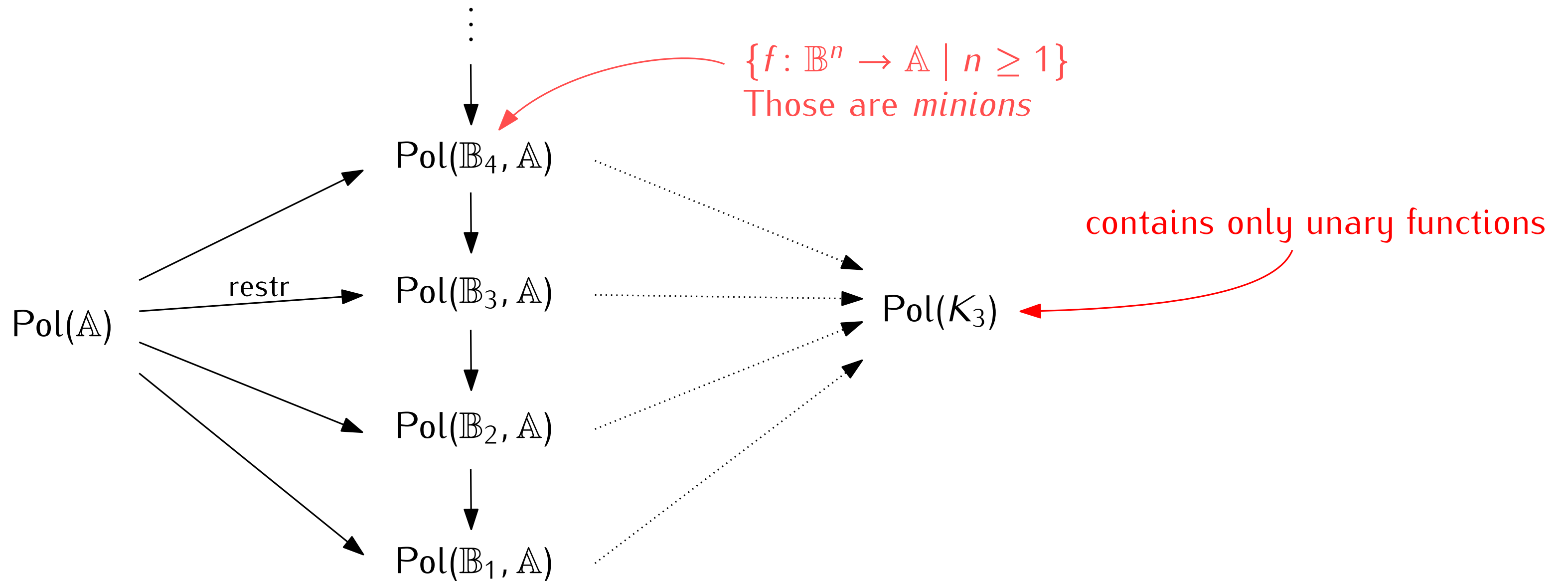
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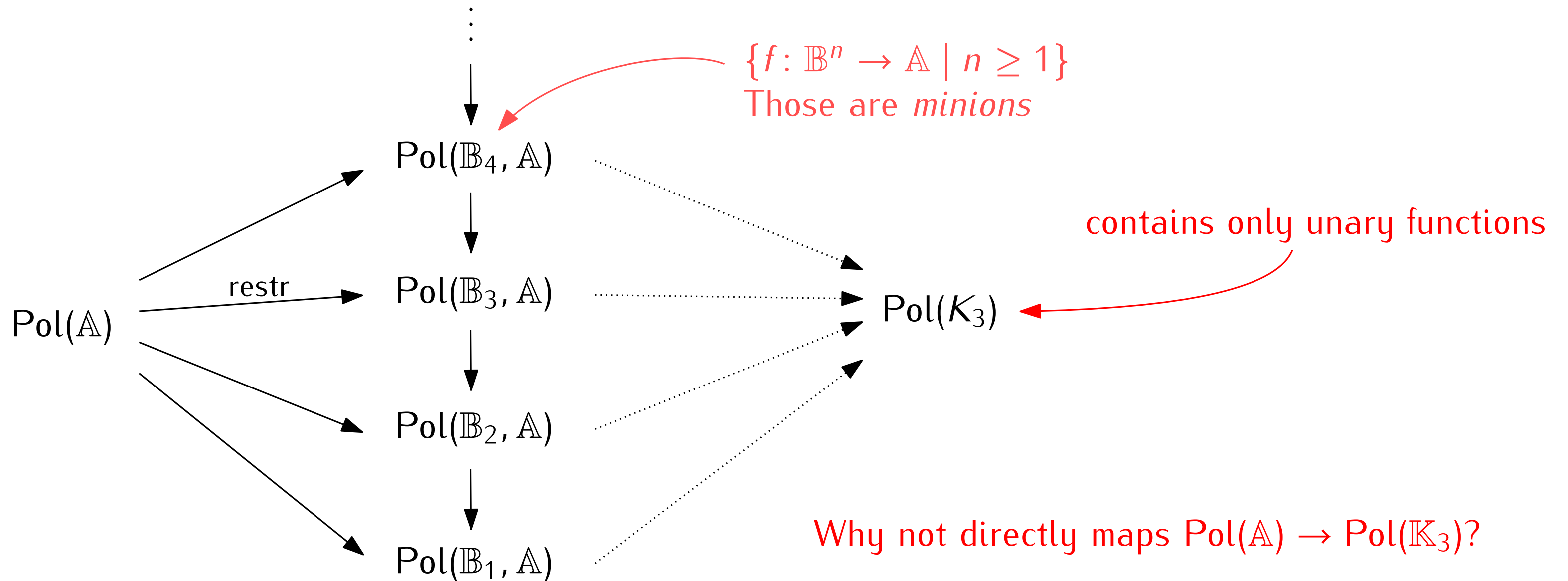
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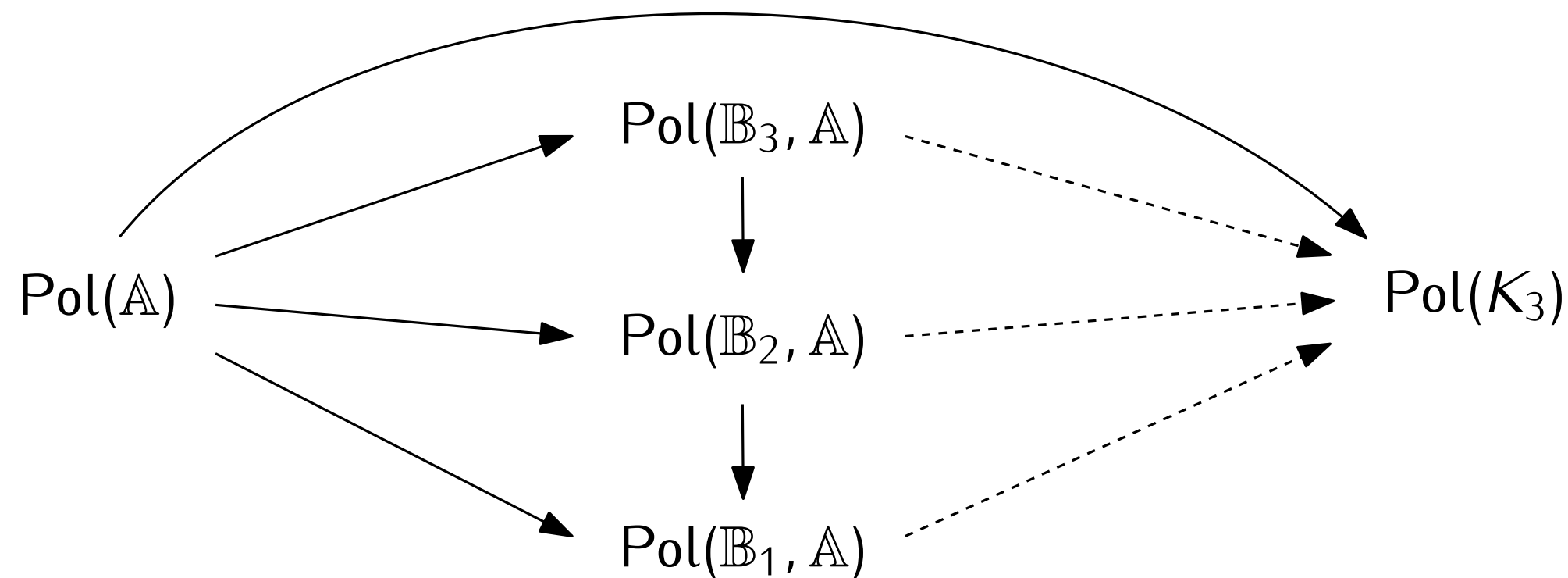
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Theorem (Gillibert, Jonušas, Kompatscher, M., Pinsker).

There exists an ω -categorical \mathbb{A} with finite signature such that:

- $\text{Pol}(\mathbb{A})$ does not satisfy any non-trivial minor condition
 $\rightsquigarrow \exists$ minion homomorphism $\text{Pol}(\mathbb{A}) \rightarrow \text{Pol}(K_3)$
- $\text{Pol}(\mathbb{B}, \mathbb{A})$ satisfies a non-trivial minor condition for every finite $\mathbb{B} \subseteq \mathbb{A}$
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\implies " $\text{CSP}(\mathbb{A}) \in \text{P}$ " expressible in existential second-order logic!

$\exists S \forall x, y, z : S(x, y, x, z, y, z) = S(y, x, z, x, z, y) \wedge S$ is a polymorphism

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We know $\text{Pol}(\mathbb{A})$ characterizes the complexity of $\text{CSP}(\mathbb{A})$, but we don't know exactly why.

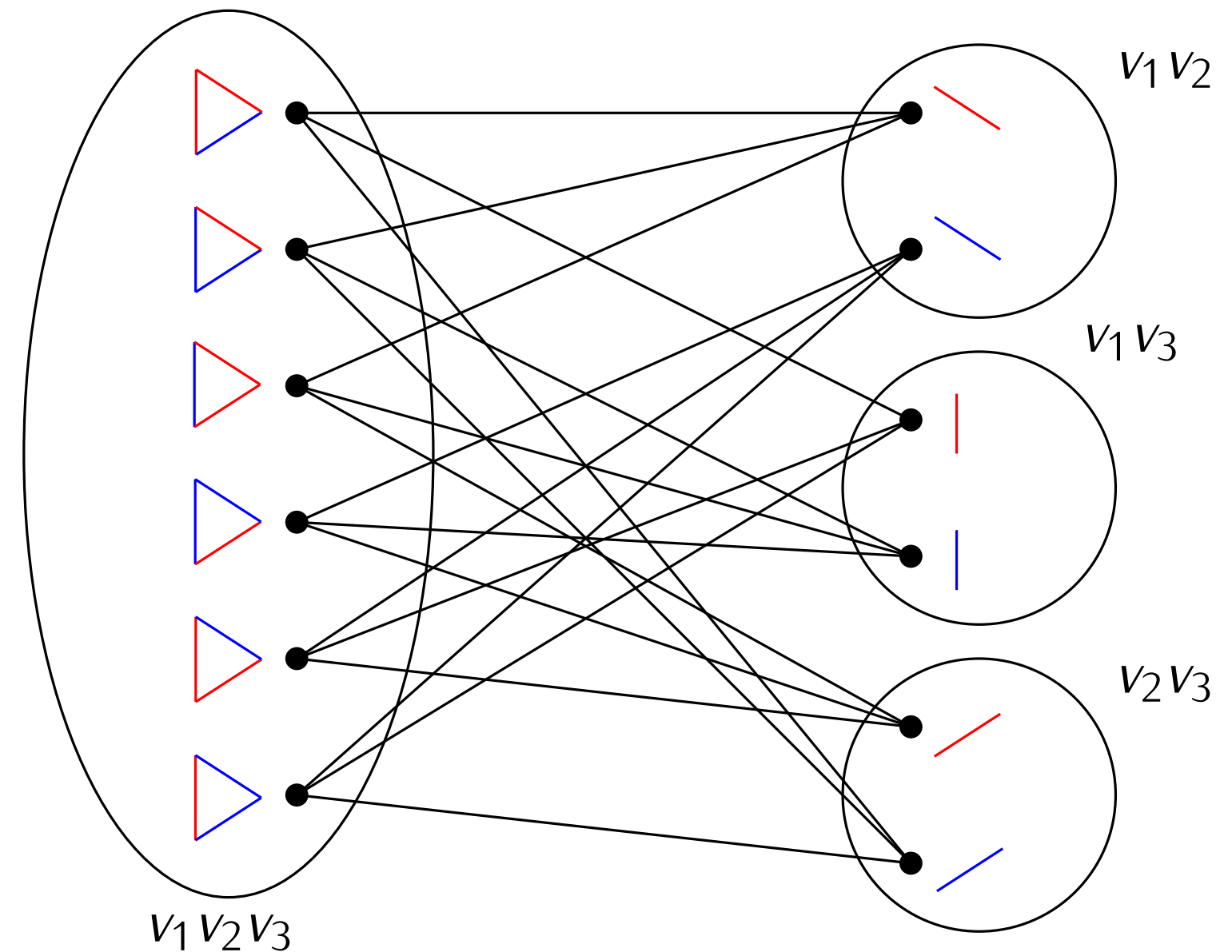
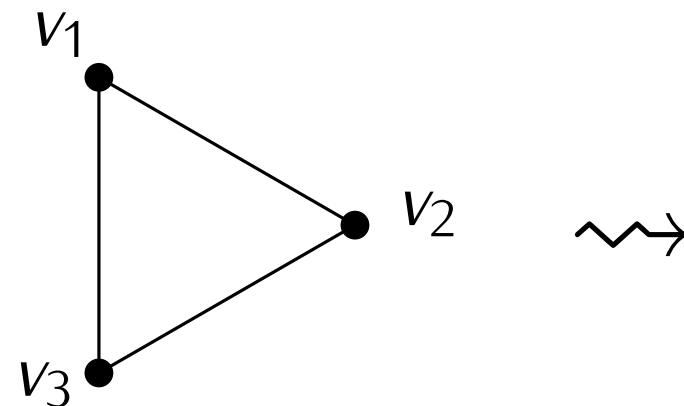
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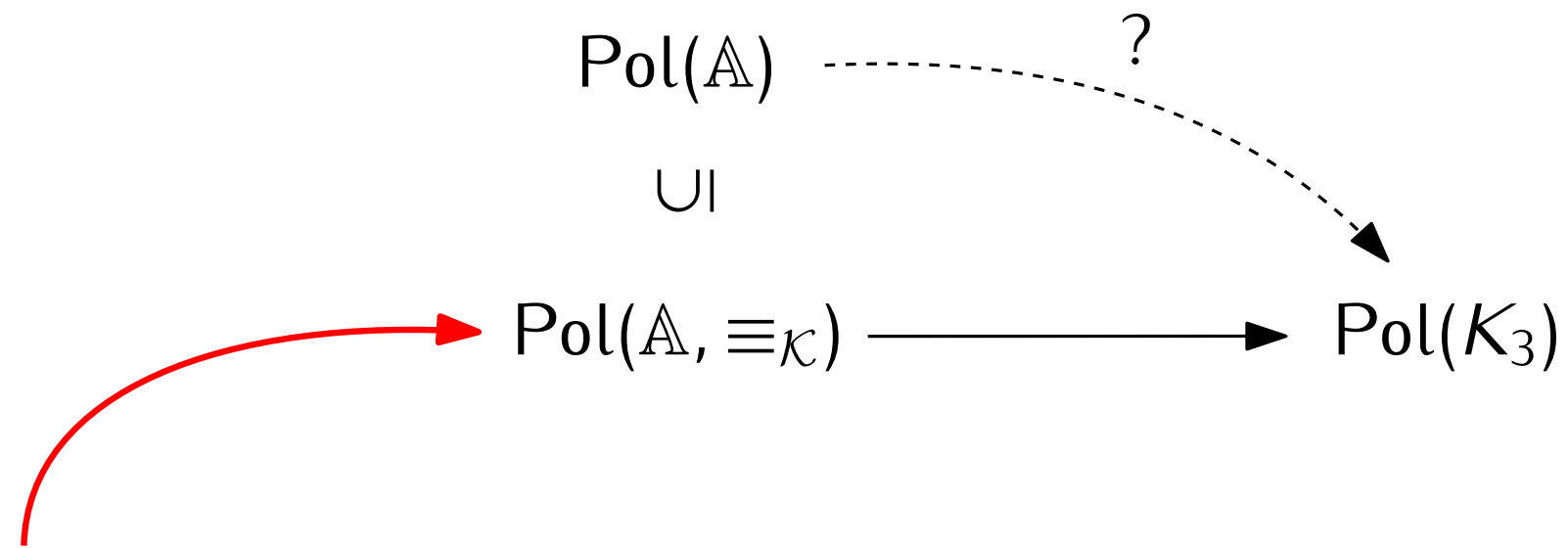
Problem (Monochromatic triangle).

- **Input:** A finite undirected graph $G = (V, E)$
- **Question:** Does there exist a coloring $\chi: E \rightarrow \{R, B\}$ such that $\chi^{-1}(c)$ is triangle-free for every $c \in \{R, B\}$?



Proving hardness of CSPs:

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- Requires extending **partial** homomorphisms $\text{Pol}(\mathbb{A}) \rightarrow \text{Pol}(K_3)$



closure properties of the
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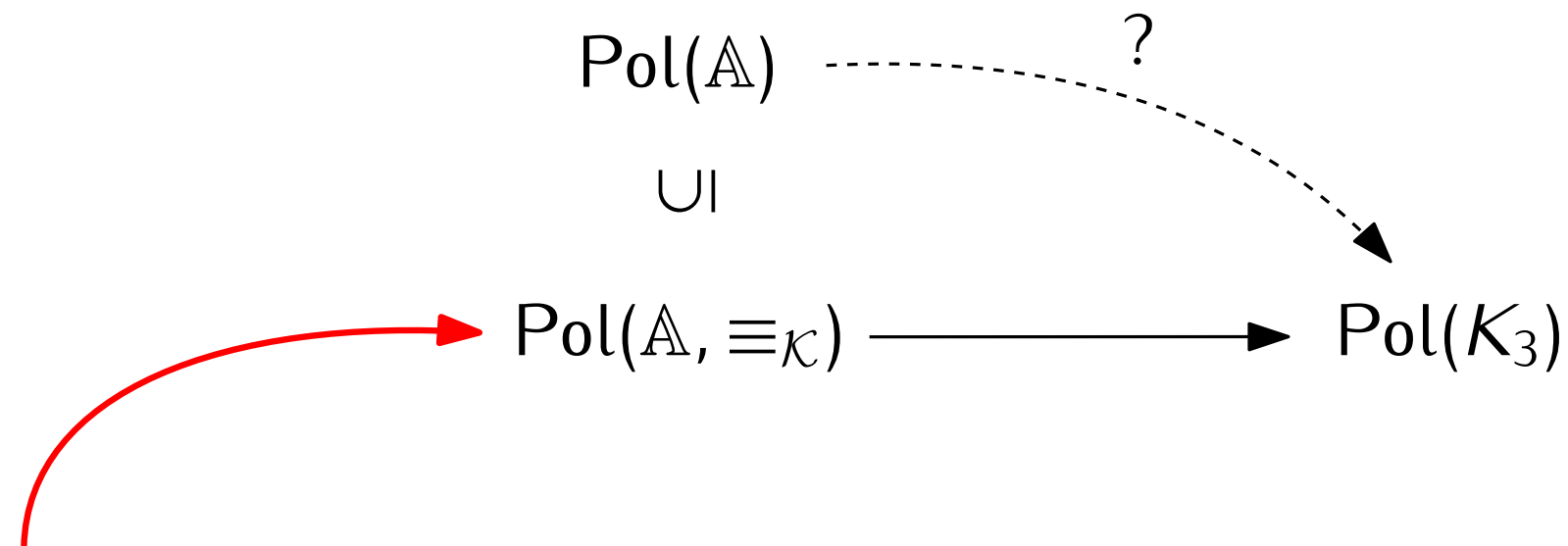
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Proving tractability of CSPs:

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Connections with **promise** CSPs

- Definability: FO-definability, descriptive complexity
- algorithmic transfers between CSPs and PCSPs
- hardness transfers from CSPs to PCSPs

(M. CSL'24)

(M. LICS'25)

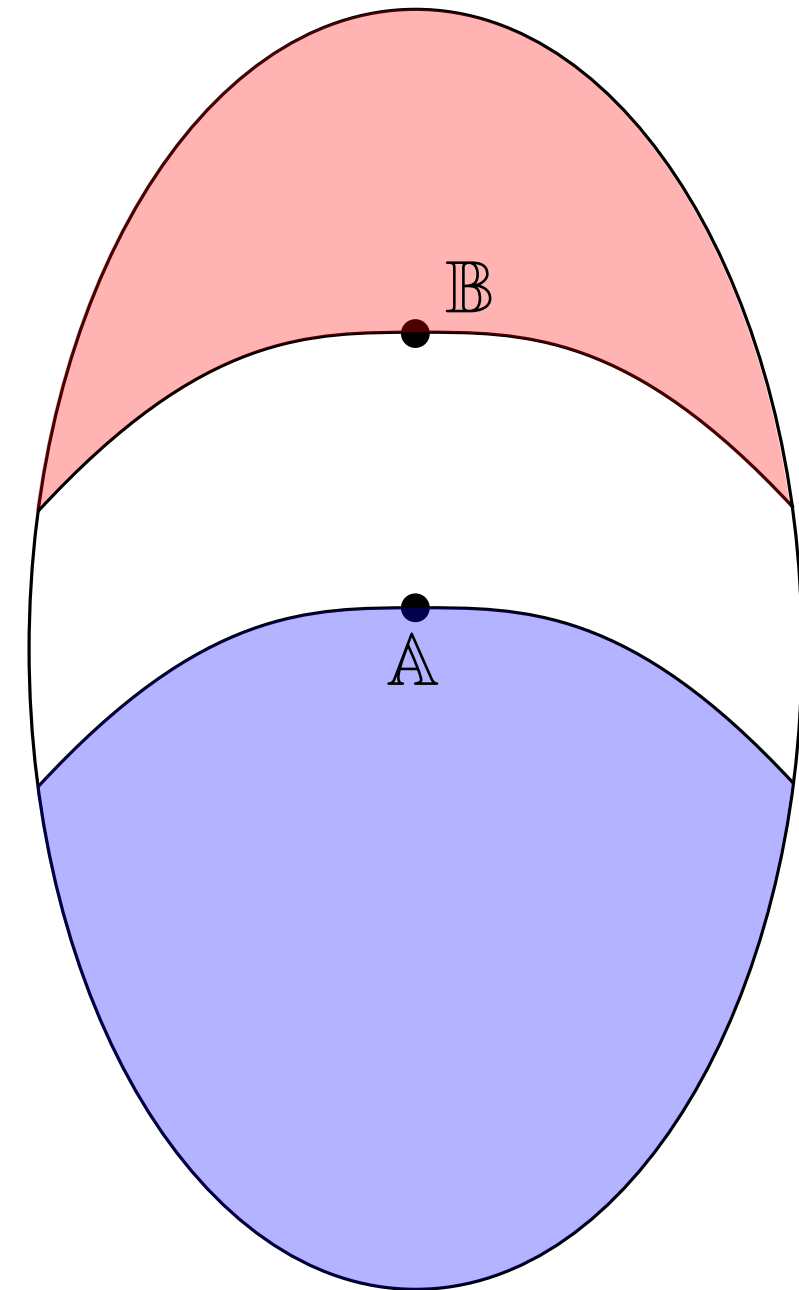
Connections with Promise CSPs

$$\mathbb{A} \rightarrow \mathbb{C} \rightarrow \mathbb{B}$$

Definition (Promise CSP). $\text{PCSP}(\mathbb{A}, \mathbb{B})$ is the problem to distinguish inputs \mathbb{X} s.t.:

Yes: $\mathbb{X} \rightarrow \mathbb{A}$

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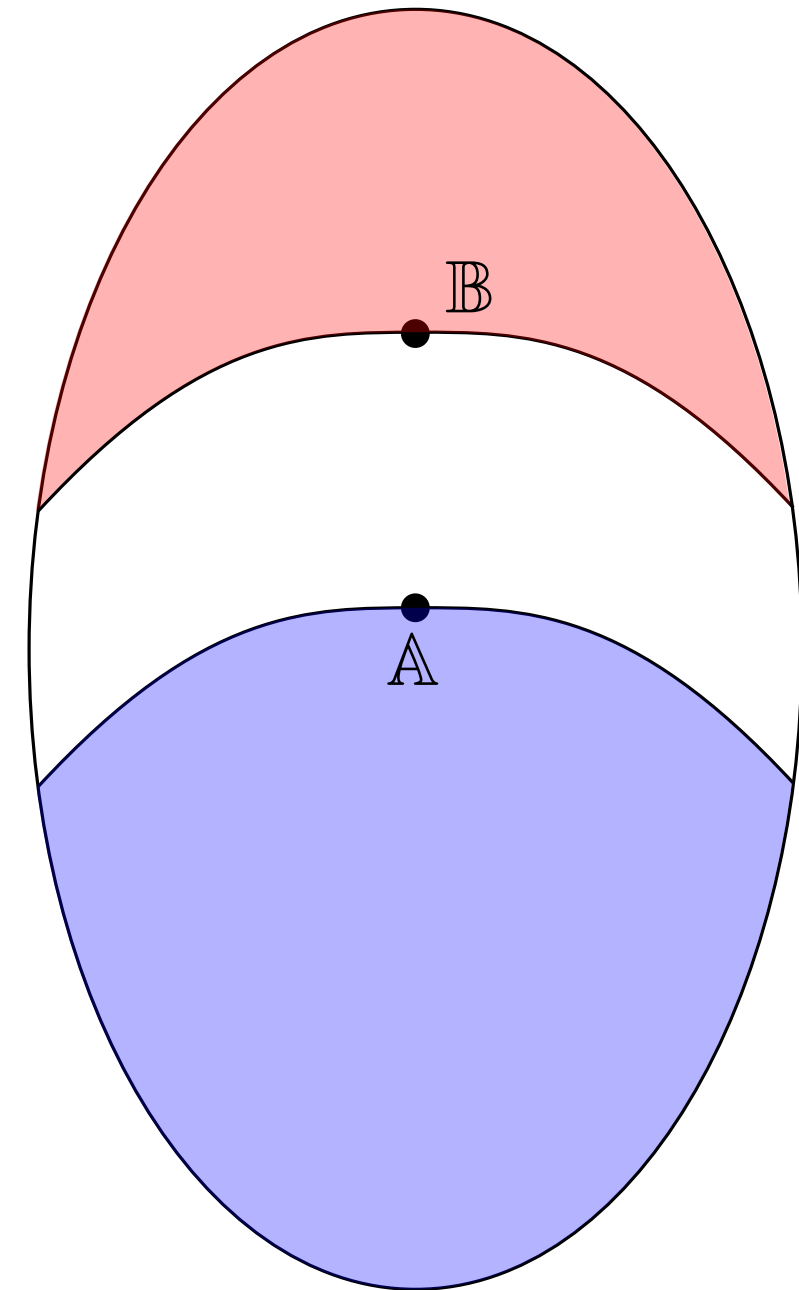
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- k -colorable $(\chi(G) \leq k)$
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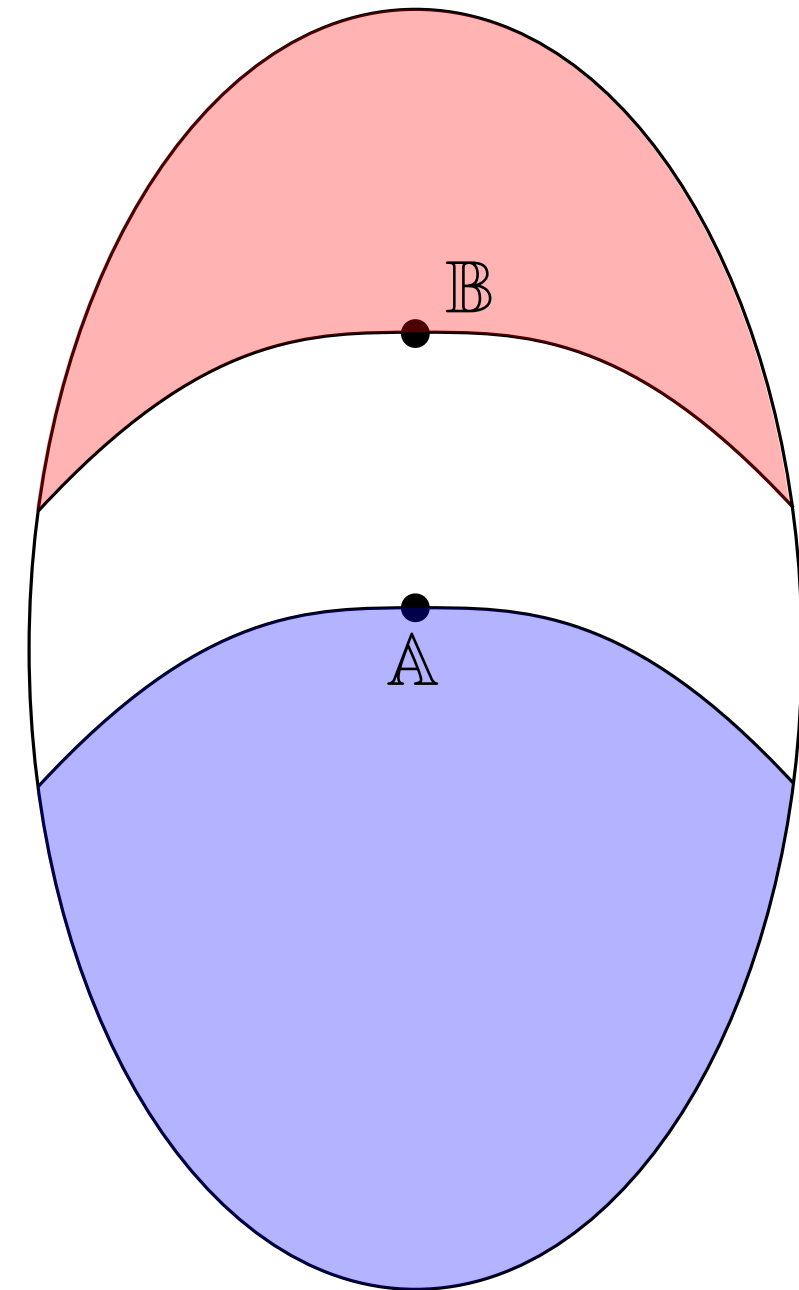
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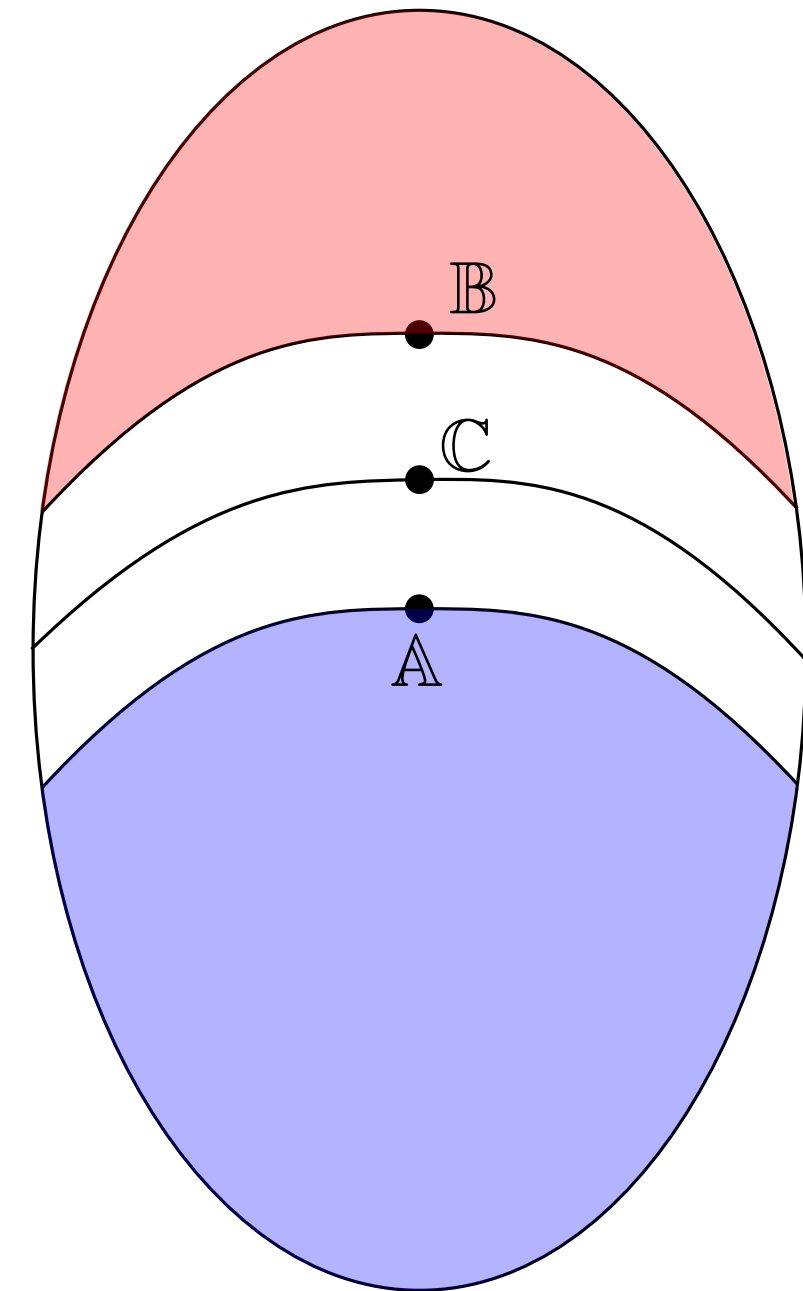
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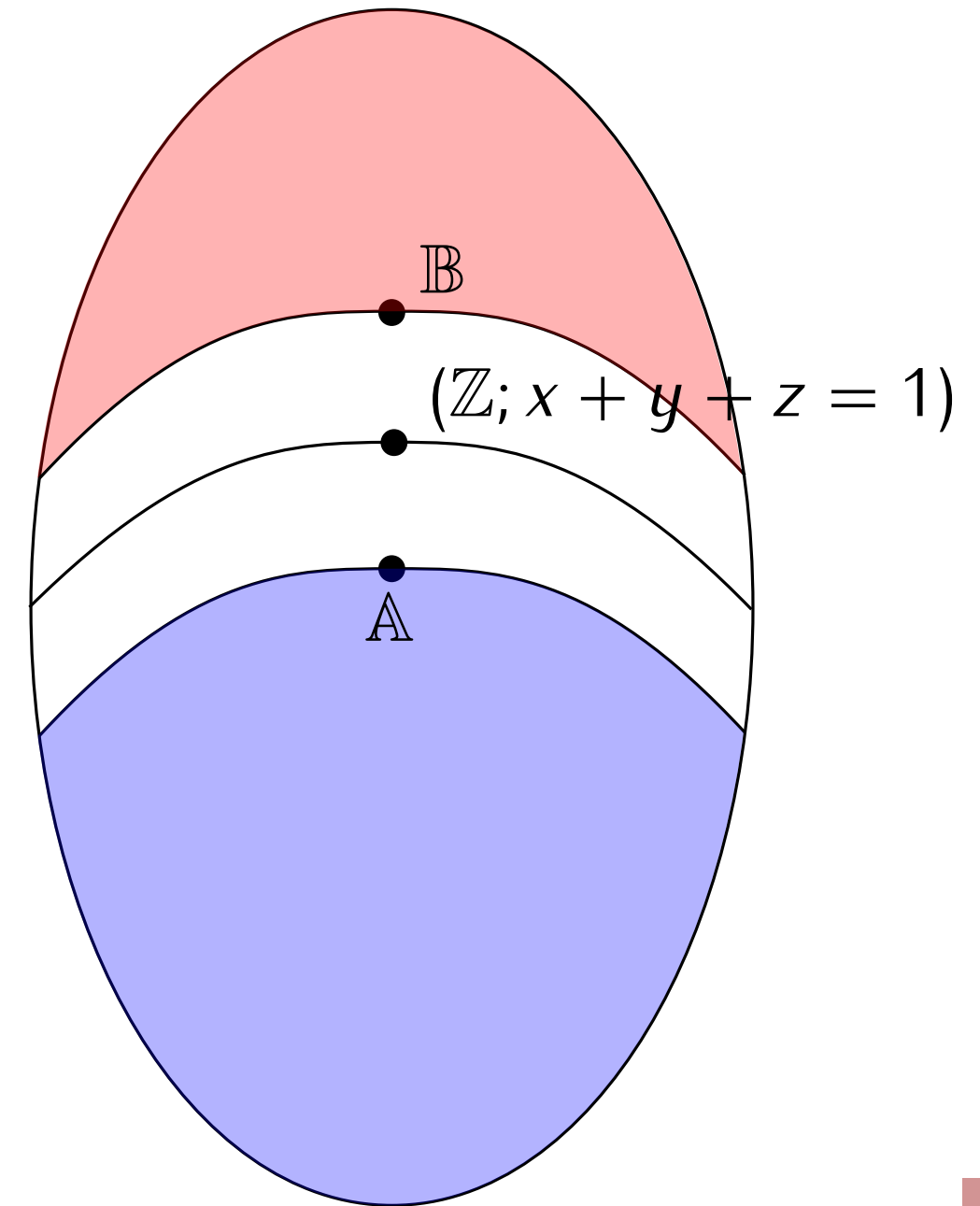
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Theorem (Brakensiek, Guruswami + Barto).

$\text{PCSP}(1in3, \text{NAE})$ is:

- solvable in polynomial time
- not solvable by a finite-domain CSP

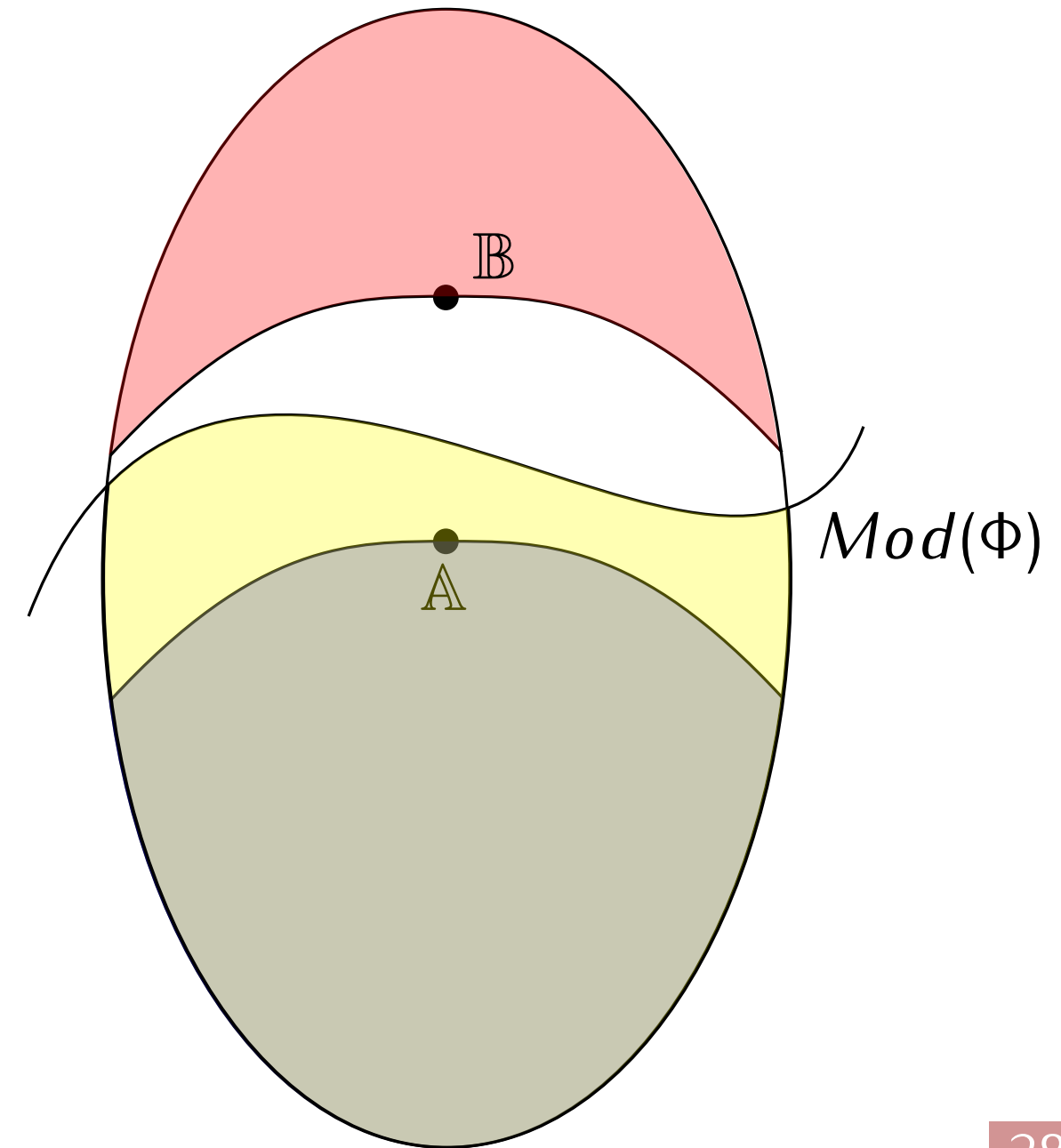


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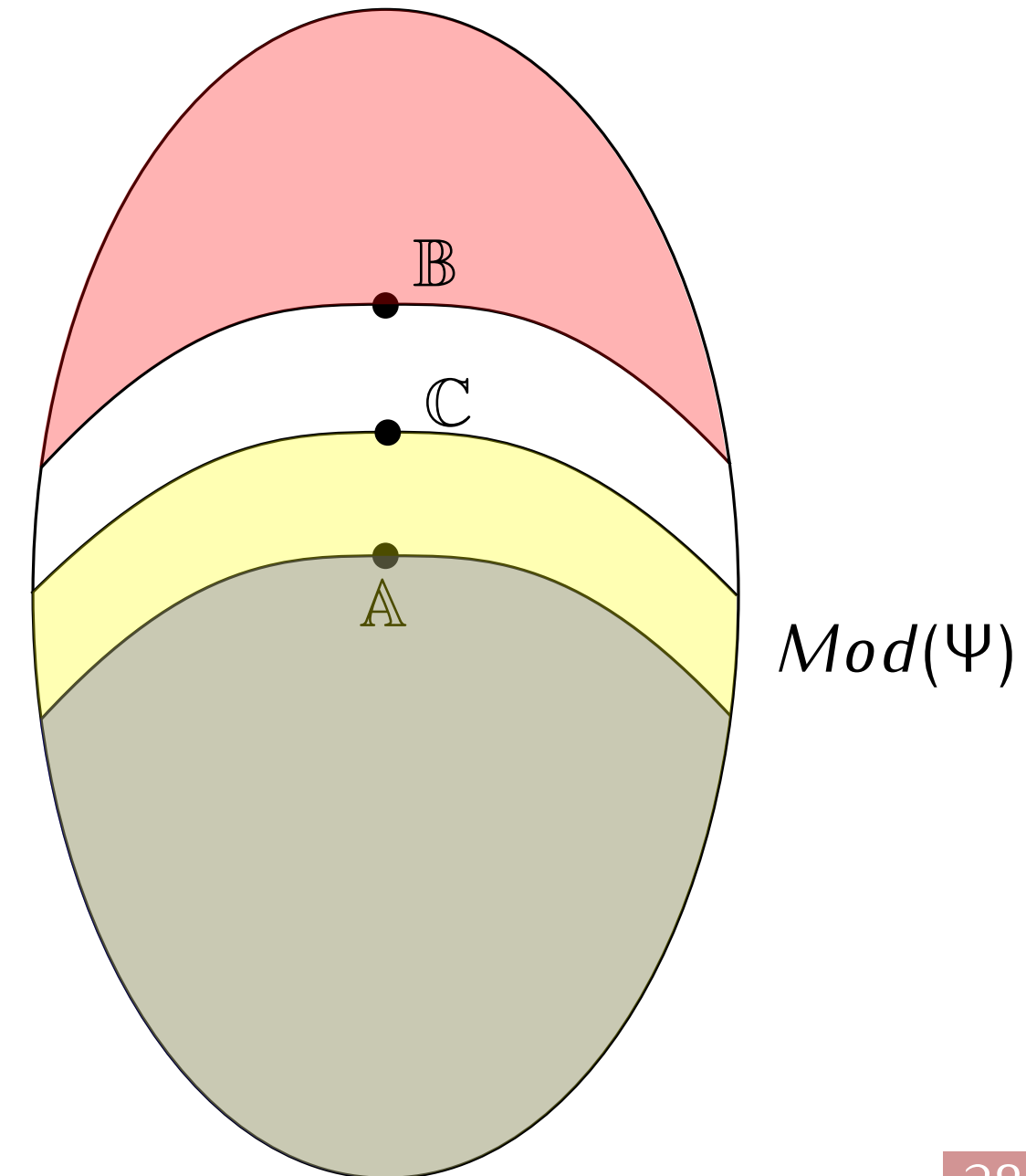
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Theorem (essentially Rossman). The following are equivalent:

- $\text{PCSP}(\mathbb{A}, \mathbb{B})$ FO-separable
- there exists a **finite** \mathbb{C} such that $\mathbb{A} \rightarrow \mathbb{C} \rightarrow \mathbb{B}$ and $\text{CSP}(\mathbb{C})$ is FO-definable



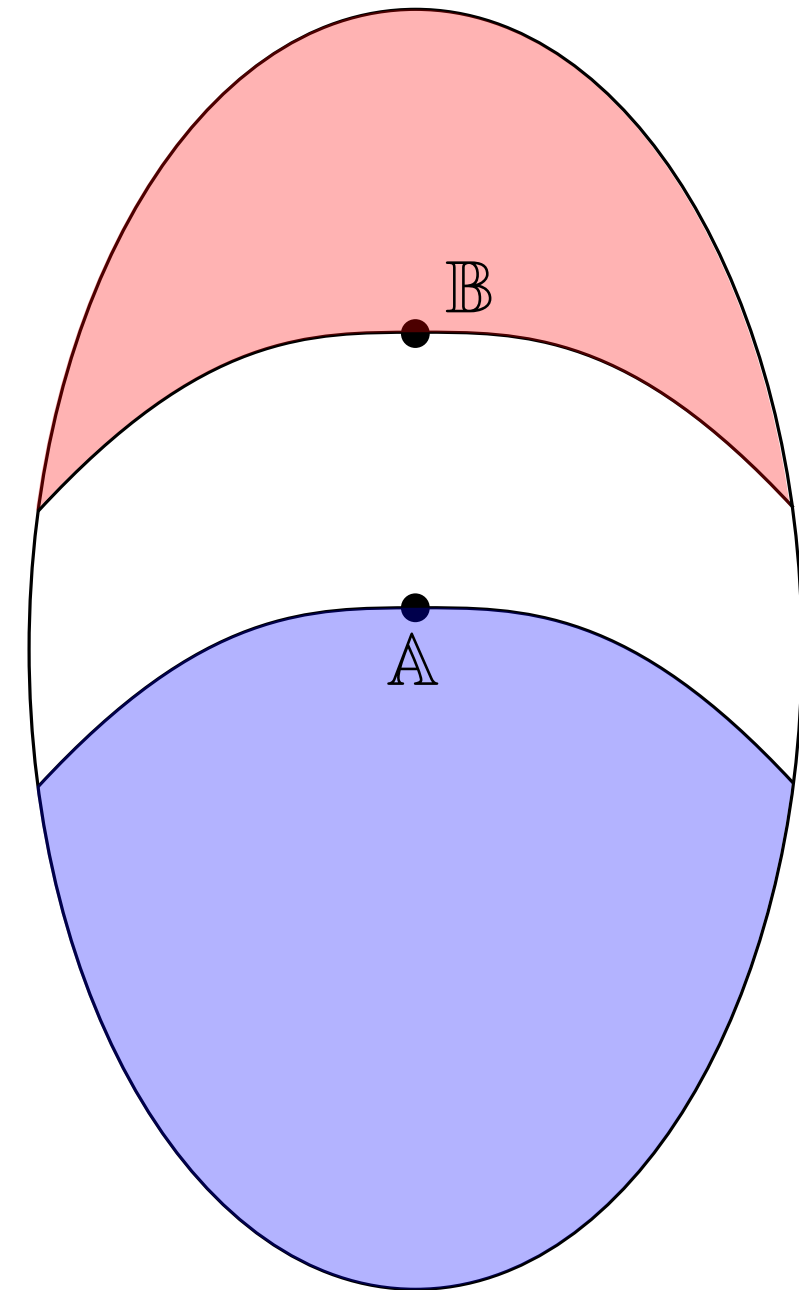
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- $\text{PCSP}(\mathbb{A}, \mathbb{B})$ solvable by a Datalog program ($\exists^+ \text{FP}$)
- not solvable by a finite-domain CSP
- solvable by a r.f.b.h. \mathbb{C} , whose CSP is not solvable by a Datalog program



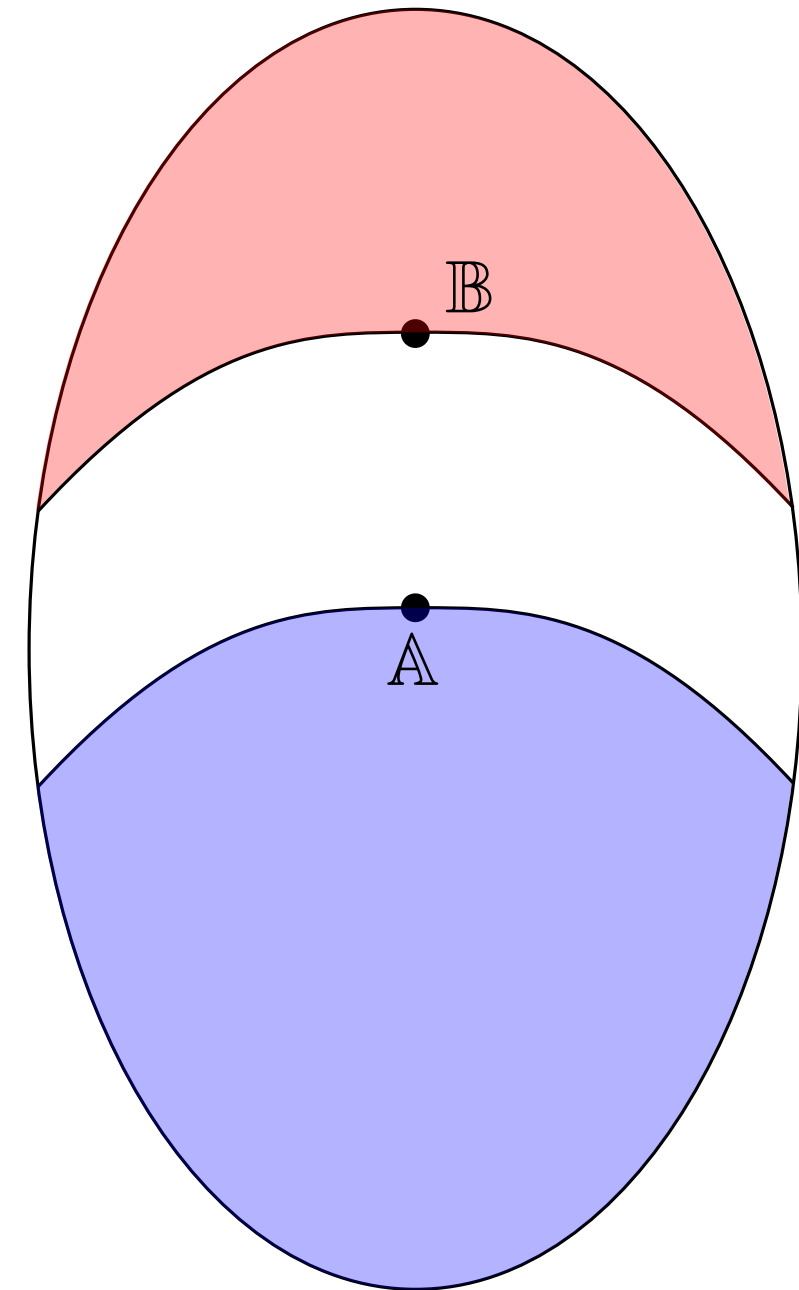
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- not solvable by a finite-domain CSP
- solvable by a r.f.b.h. \mathbb{C}



Definition (Promise CSP). $\text{PCSP}(\mathbb{A}, \mathbb{B})$ is the problem to distinguish inputs \mathbb{X} s.t.:

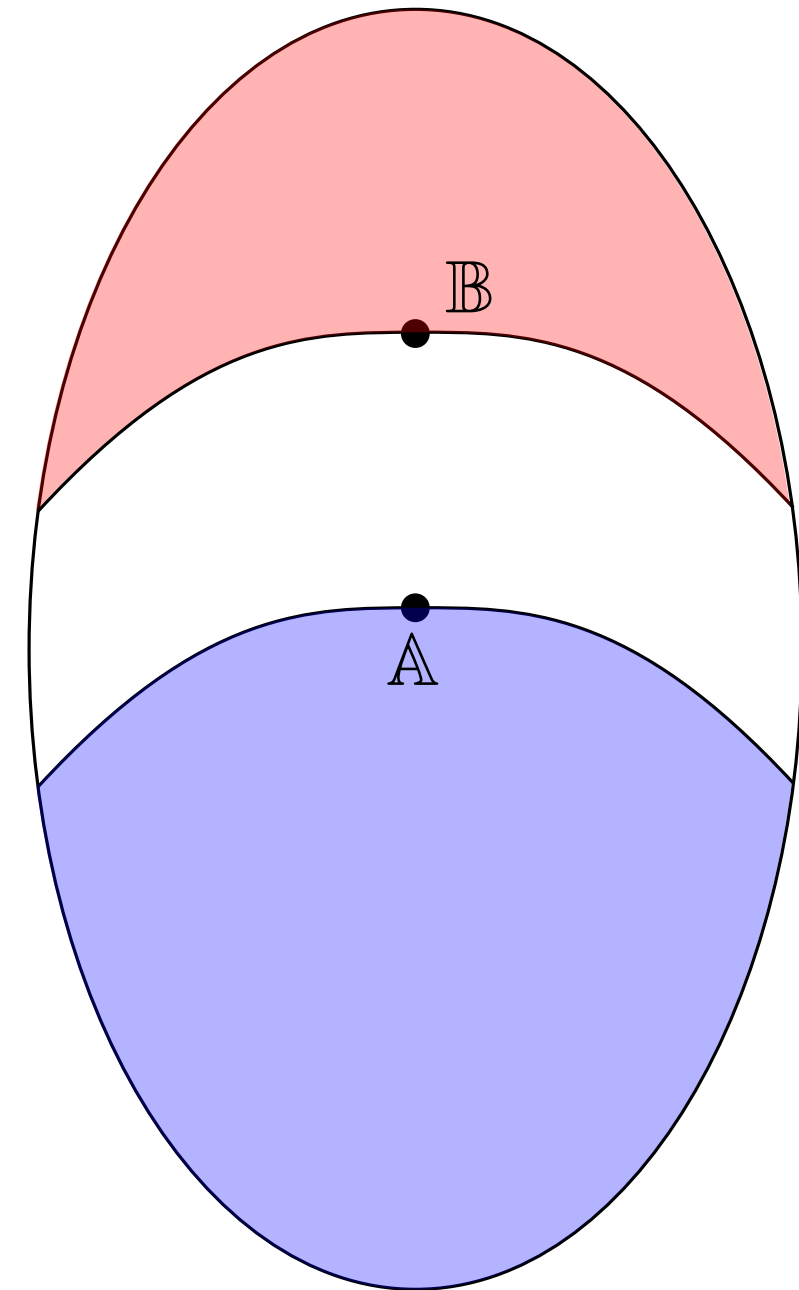
Yes: $\mathbb{X} \rightarrow \mathbb{A}$

No: $\mathbb{X} \not\rightarrow \mathbb{B}$

Theorem (M.). There exist finite \mathbb{A}, \mathbb{B} such that:

- $\text{PCSP}(\mathbb{A}, \mathbb{B})$ is not solvable in FPC
- not solvable by a finite-domain CSP
- solvable by a r.f.b.h. \mathbb{C}

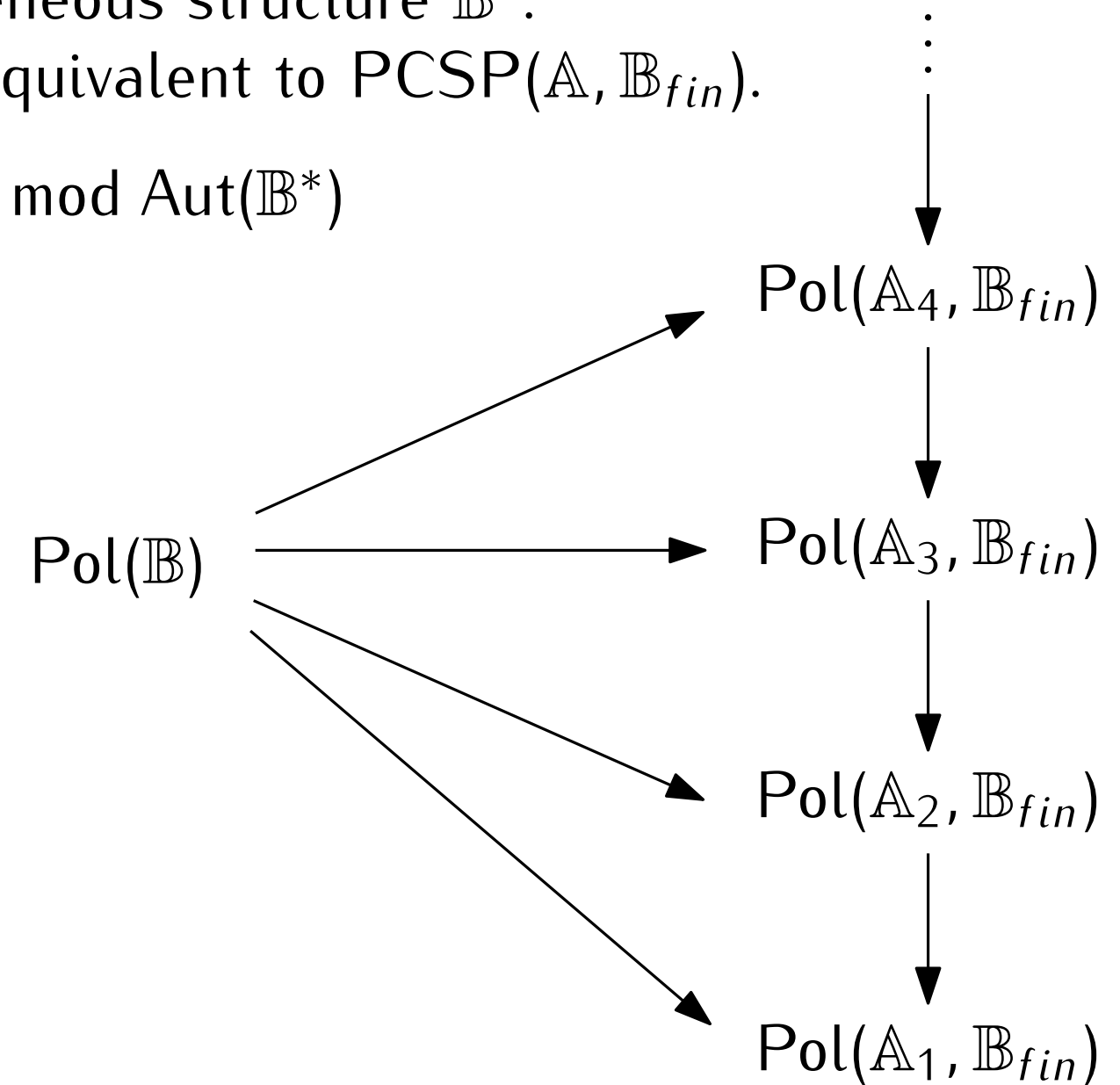
Many questions related to logical separability remain open



Theorem (M.). Let \mathbb{B} be a reduct of a finitely bounded homogeneous structure \mathbb{B}^* .
There exists \mathbb{A} (infinite) and \mathbb{B}_{fin} (finite) s.t. $\text{CSP}(\mathbb{B})$ is ptime equivalent to $\text{PCSP}(\mathbb{A}, \mathbb{B}_{fin})$.

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$$(\forall \mathbb{A}' \subset \mathbb{A} \text{ finite} : \text{Pol}(\mathbb{A}', \mathbb{B}_{fin}) \models \Sigma) \iff \text{Pol}(\mathbb{B}) \models \Sigma \text{ mod Aut}(\mathbb{B}^*)$$



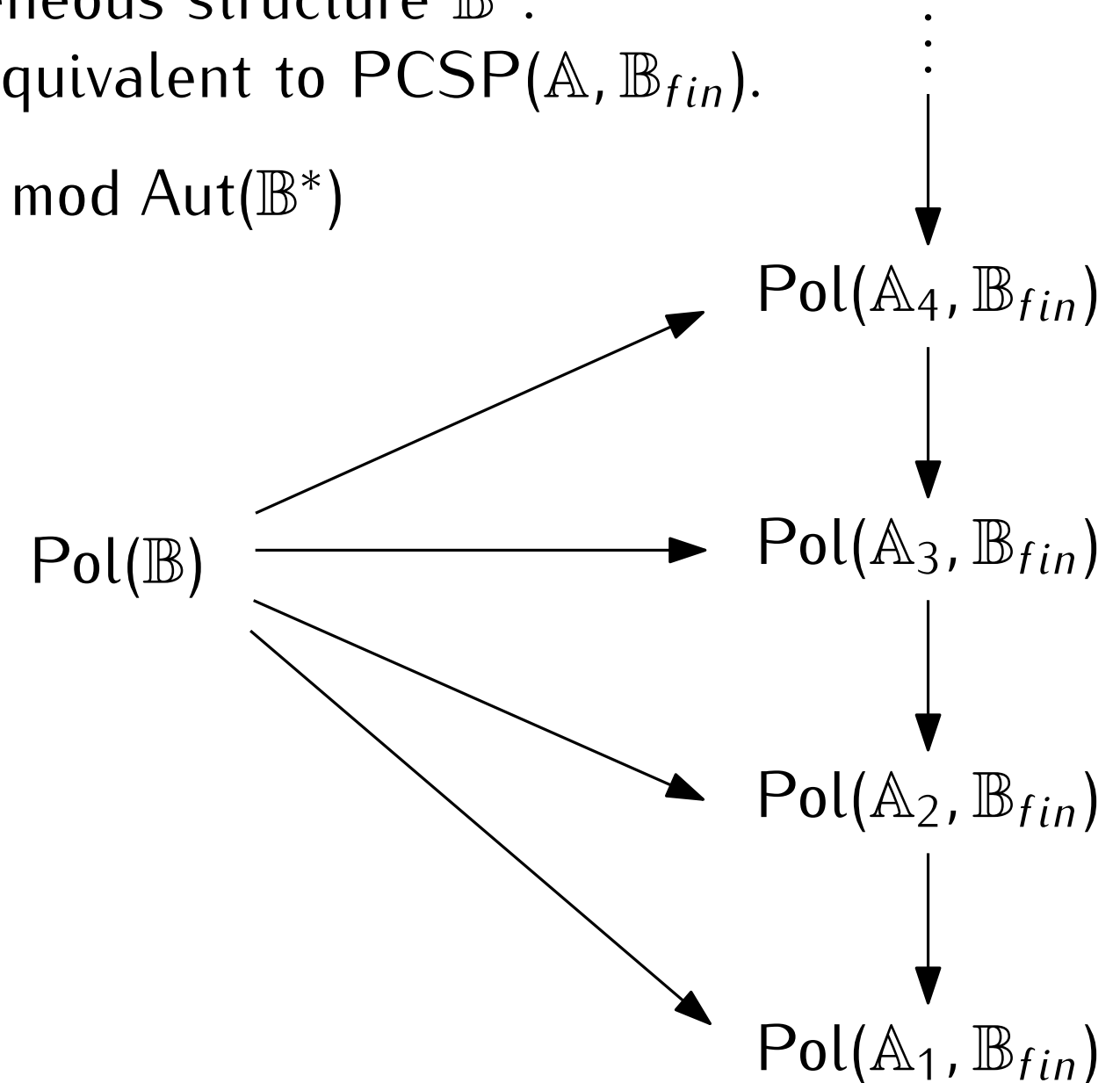
Finite minions!

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Corollary (Barto, Bulín, Opršal, Krokhin).
If $\text{Pol}(\mathbb{B})$ does not contain pseudo-Olšák operations, then $\text{CSP}(\mathbb{B})$ is NP-hard.

$$e_1 f(x, x, y, y, y, x) \approx e_2 f(x, y, x, y, x, y) \approx e_3 f(y, x, x, x, y, y)$$



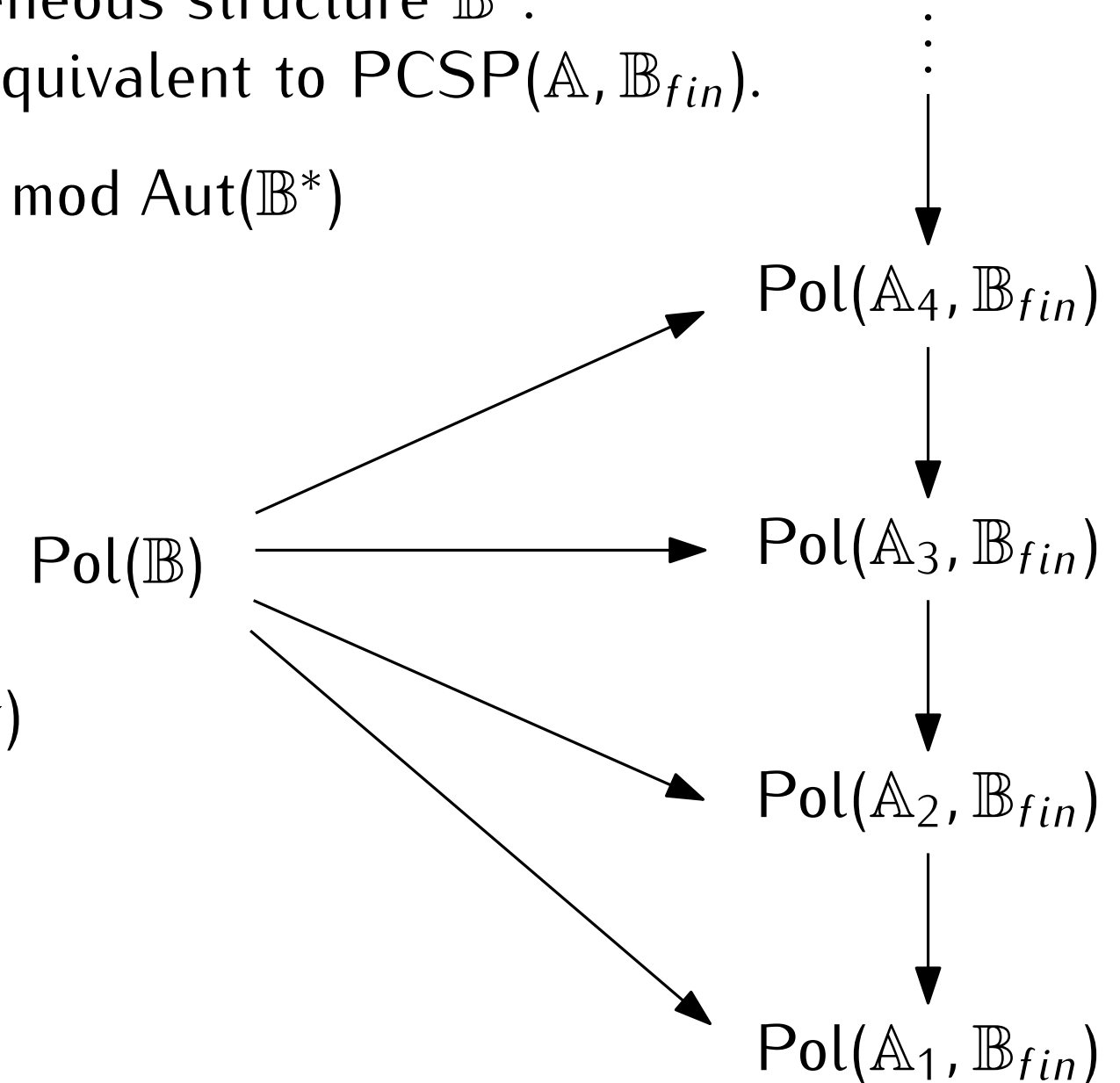
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Corollary (Atserias, Dalmau). If $\text{Pol}(\mathbb{B})$ does not contain a pseudo-WNU of some arity $k \geq 3$, then $\text{CSP}(\mathbb{B})$ is not in Datalog.

$$e_1 w(x, \dots, x, x, y) \approx e_2 w(x, \dots, x, y, x) \approx \dots \approx e_k w(y, x, \dots, x)$$



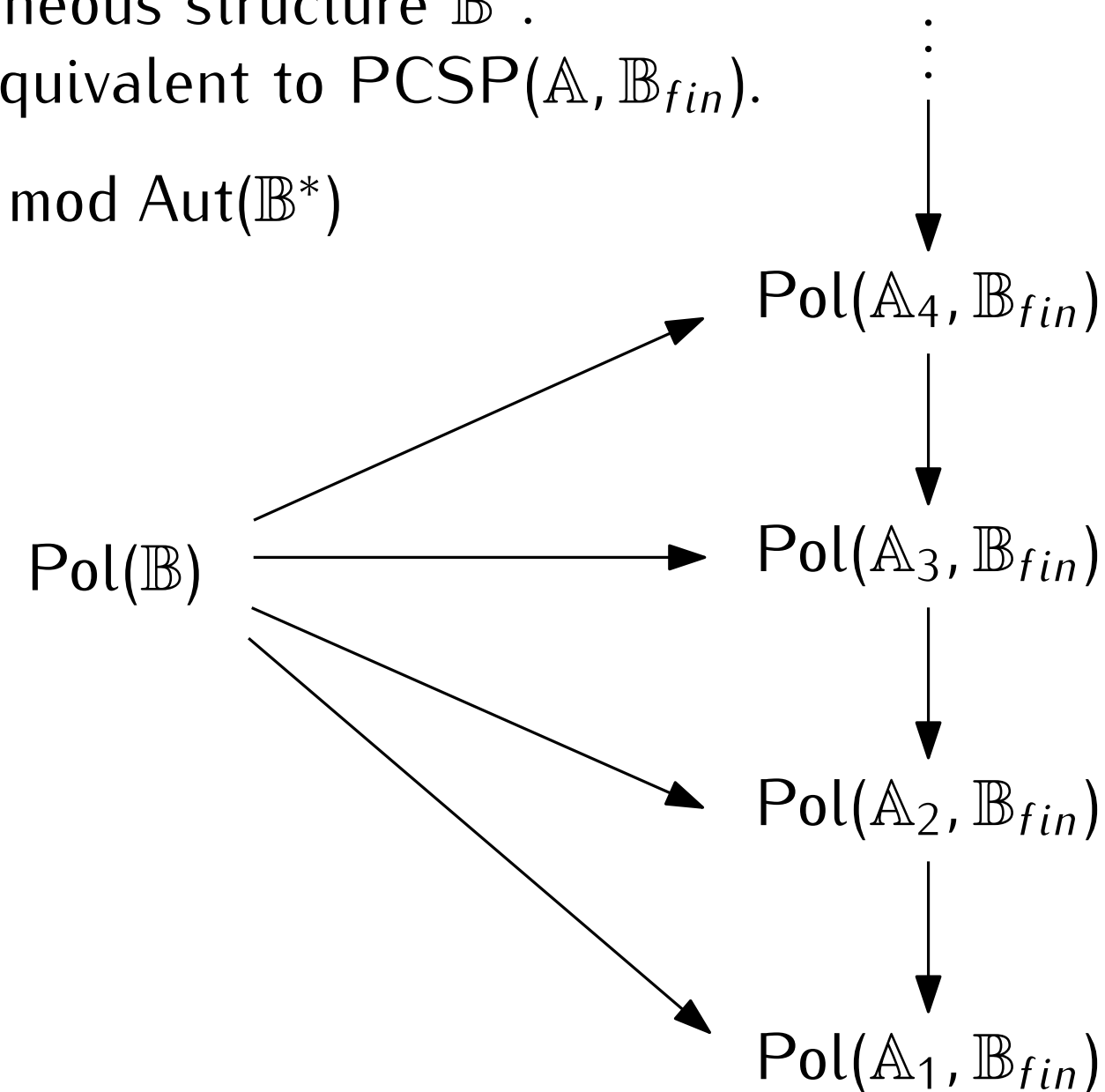
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Corollary (Barto, Bulín, Opršal, Krokhin). If $\text{Pol}(\mathbb{B})$ does not contain a pseudo-Siggers operation, then $\text{PCSP}(\mathbb{K}_3, \mathbb{K}_r)$ reduces to $\text{CSP}(\mathbb{B})$.

$$u \circ s(x, y, x, z, y, z) \approx v \circ s(y, x, z, x, z, y)$$



Finite minions!

Problem (Monochromatic triangle). Following decision problem:

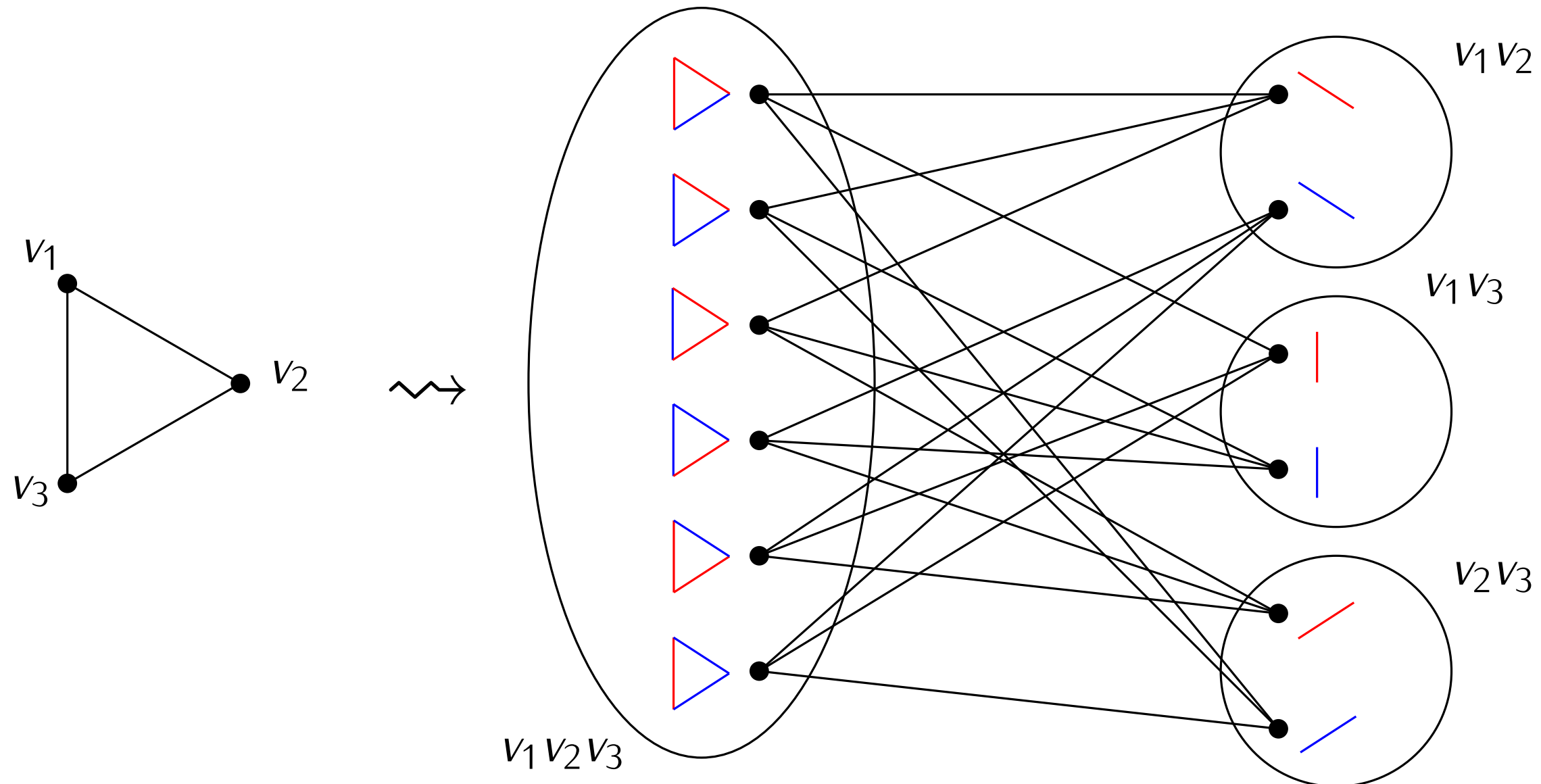
Input A finite undirected graph $G = (V, E)$

Question Does there exist a coloring $\chi: E \rightarrow \{R, B\}$ such that $\chi^{-1}(c)$ is triangle-free for every $c \in \{R, B\}$?

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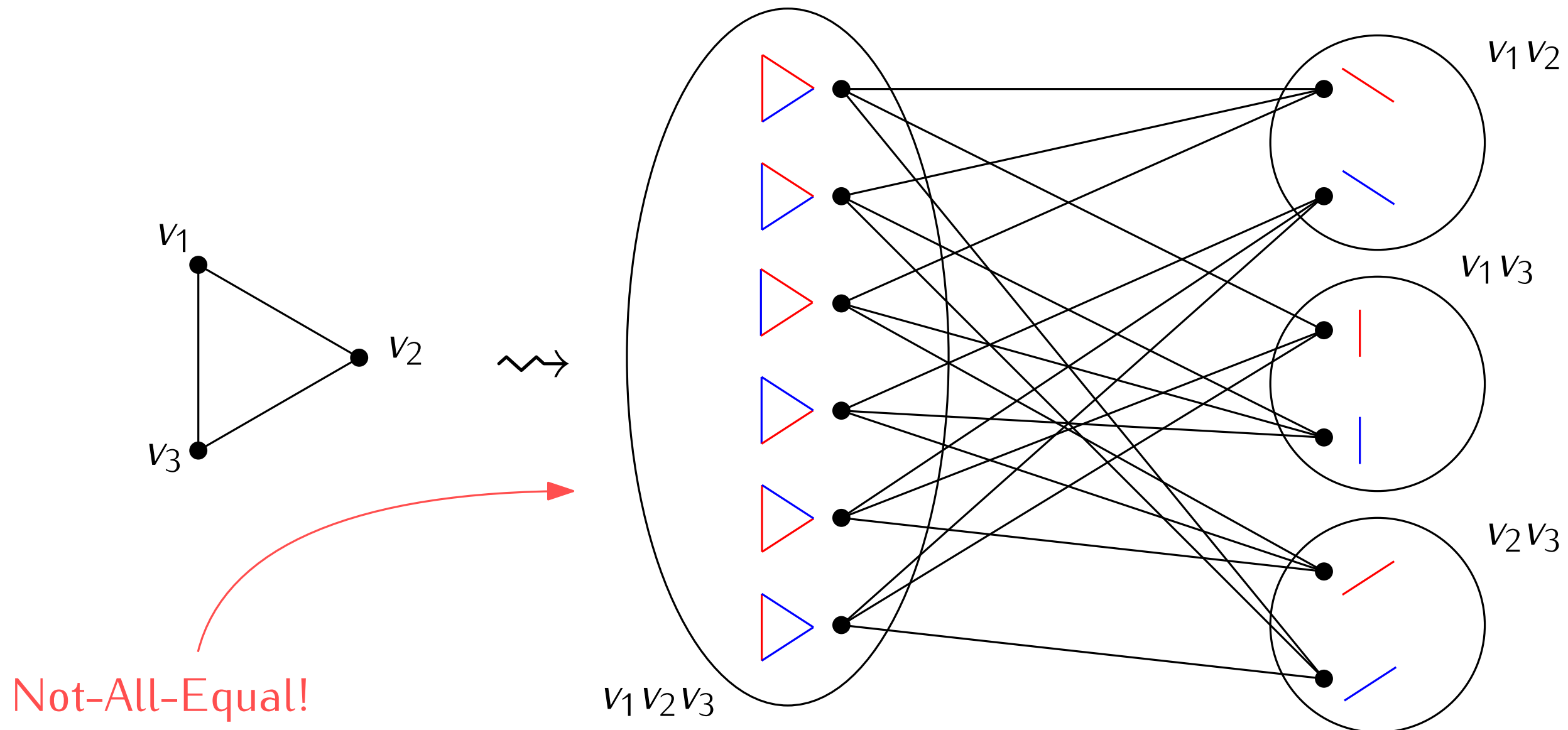
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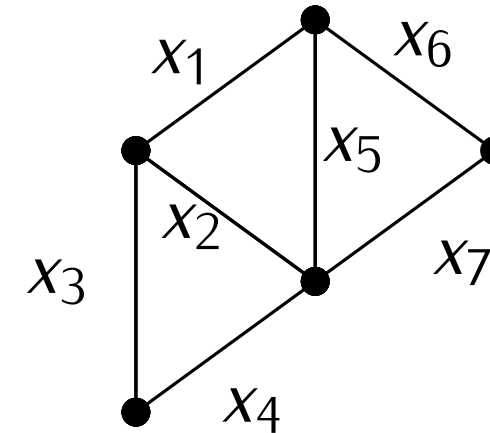
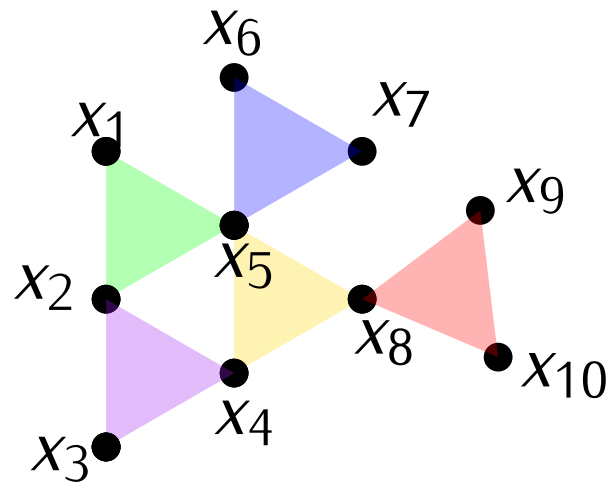


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First attempt:



Instance of Not-All-Equal

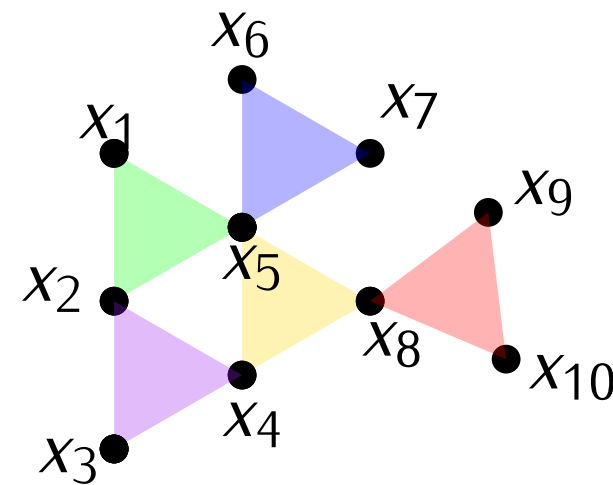
Instance of monochromatic triangle

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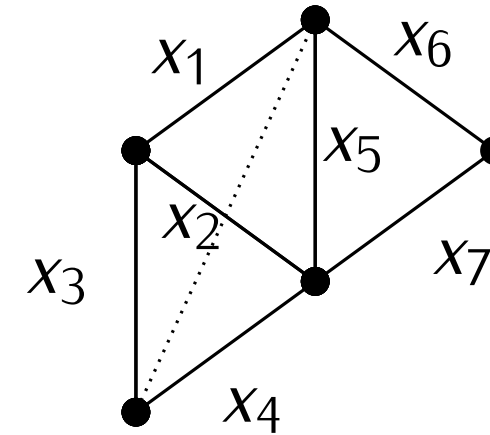
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Instance of Not-All-Equal



Where to put the edge for x_8 ?

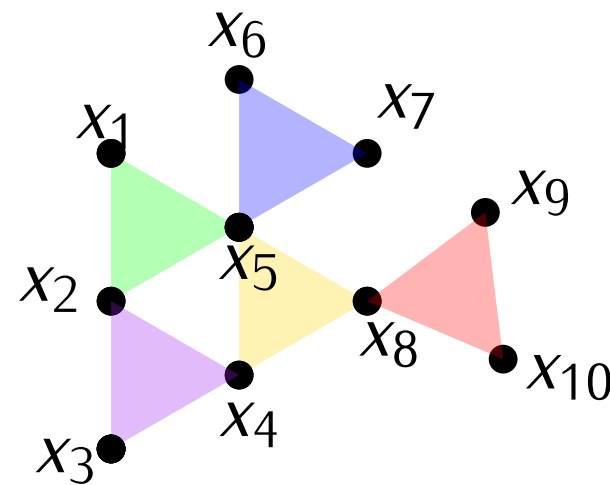
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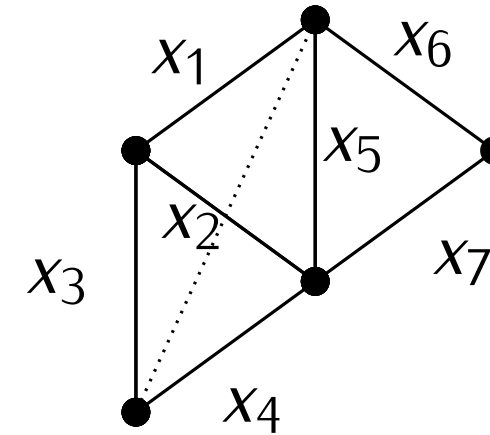
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\rightsquigarrow



Where to put the edge for x_8 ?

Instance of Not-All-Equal

Instance of monochromatic triangle

The difference between NAE and Monochromatic Triangle:

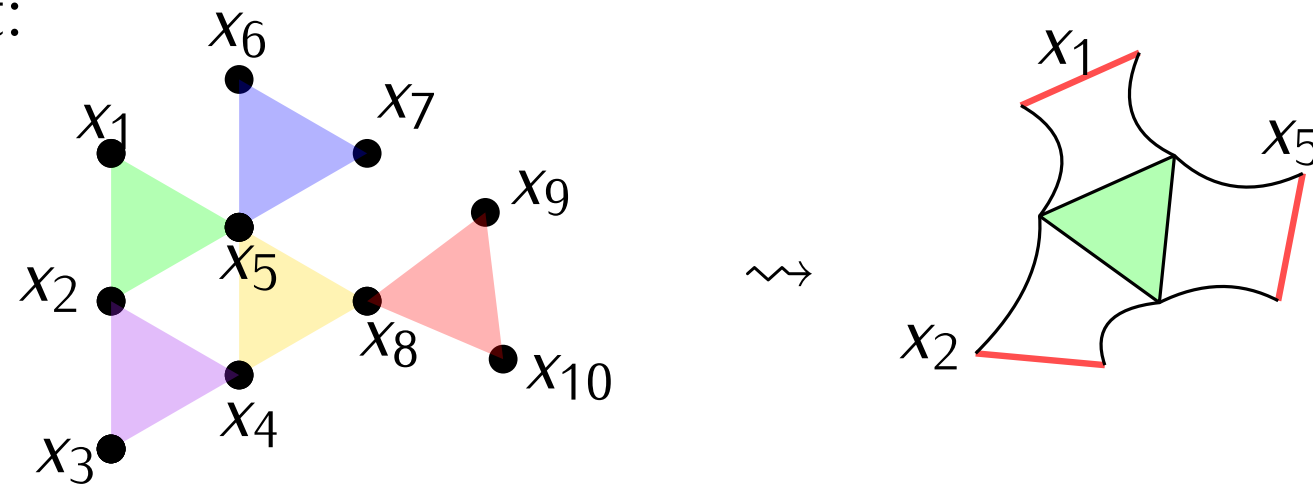
- In NAE: unrestricted use of ternary constraints
- In Monochromatic triangle: ternary constraints (on edges) can only be imposed on triangles, and are **automatically** imposed on triangles

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Instance of Not-All-Equal

The difference between NAE and Monochromatic Triangle:

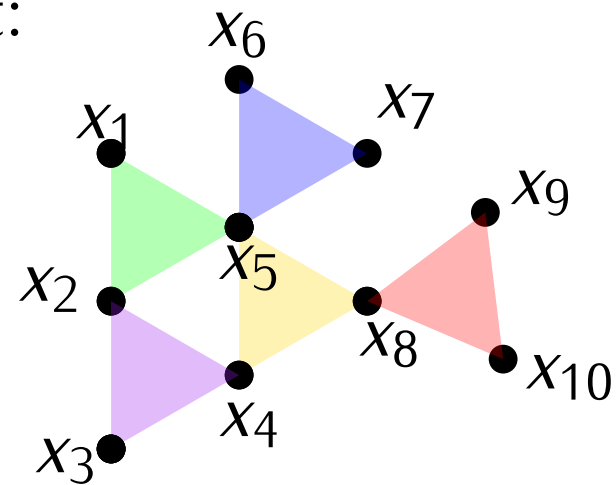
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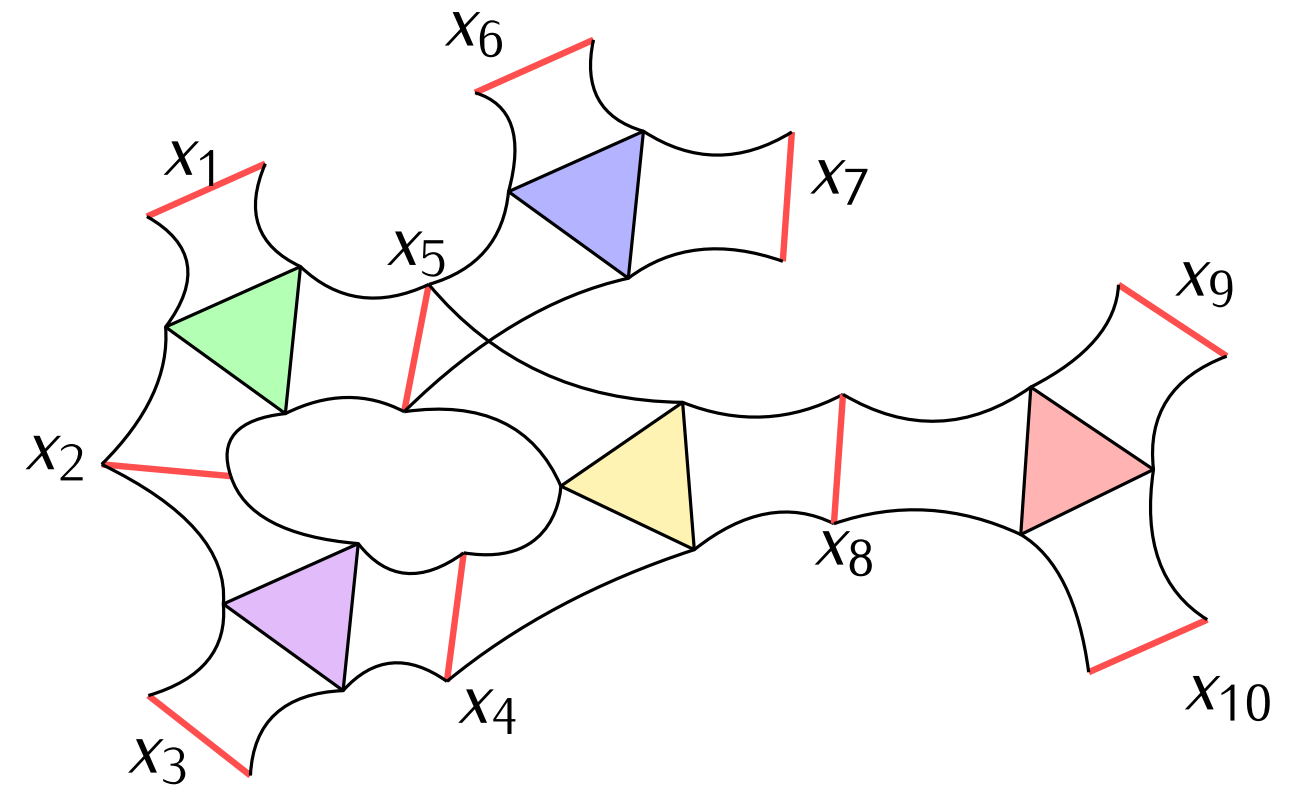
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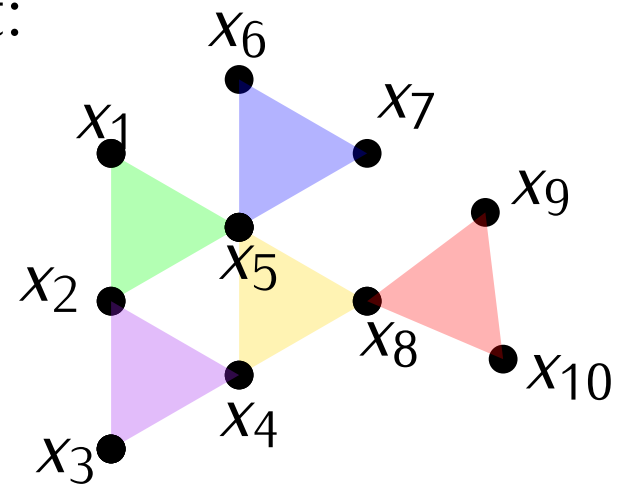
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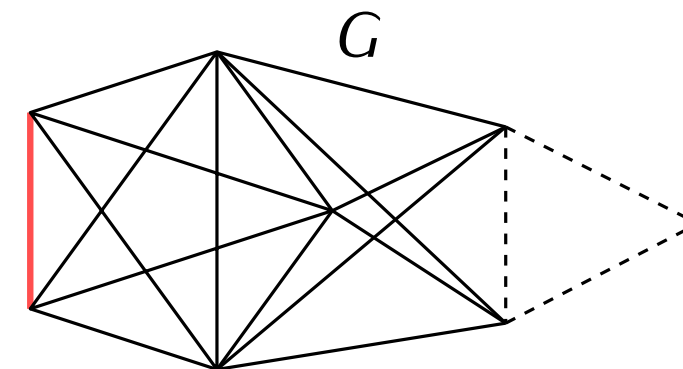
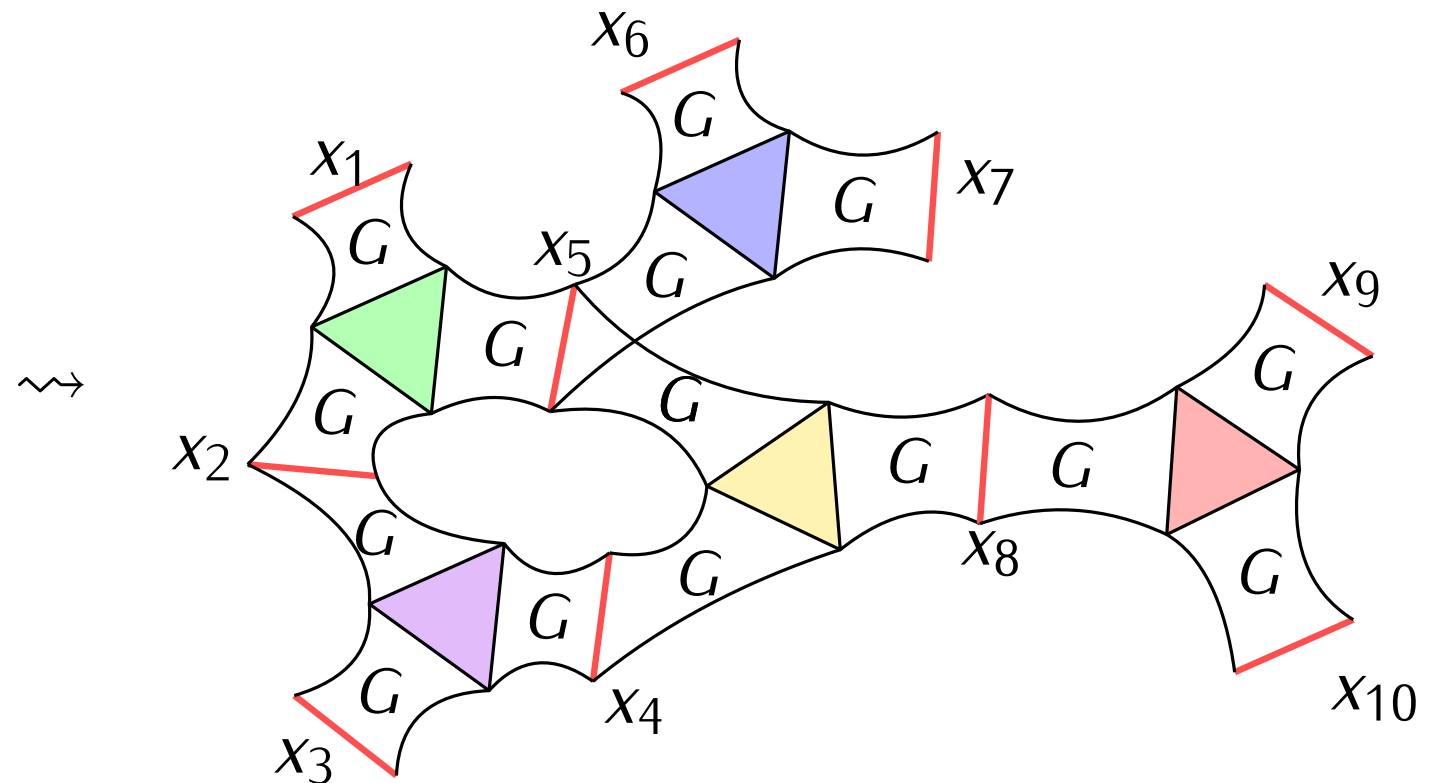
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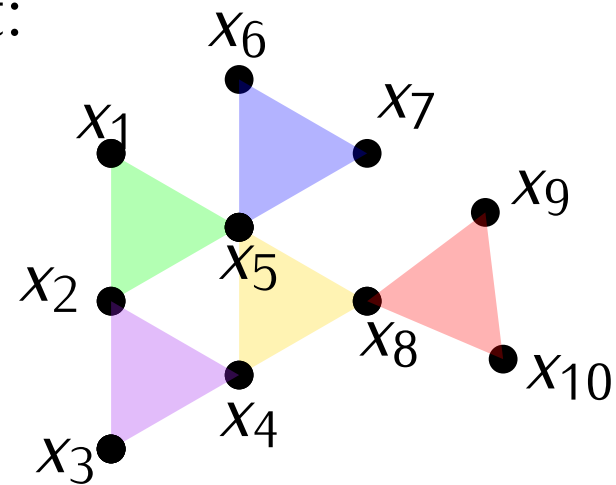


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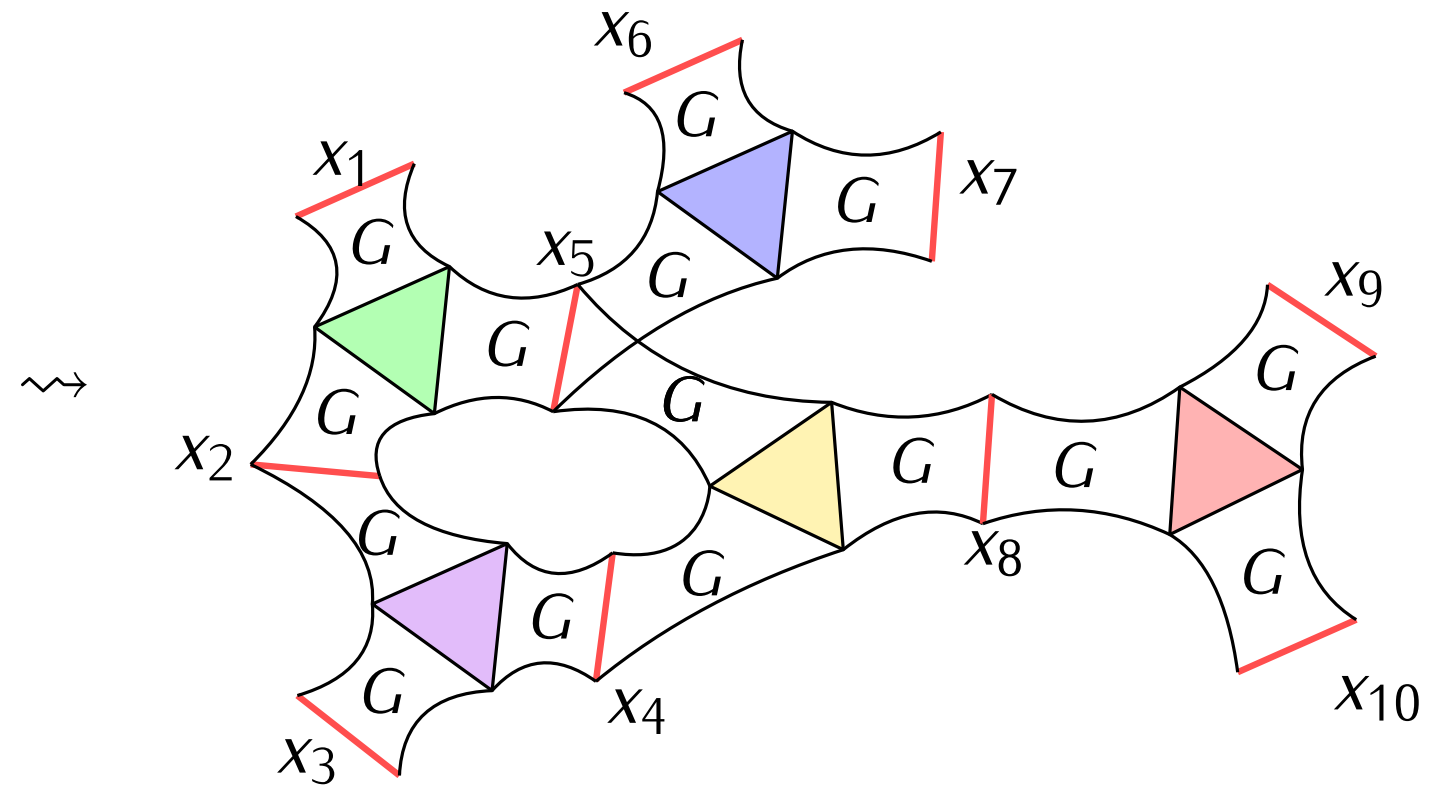
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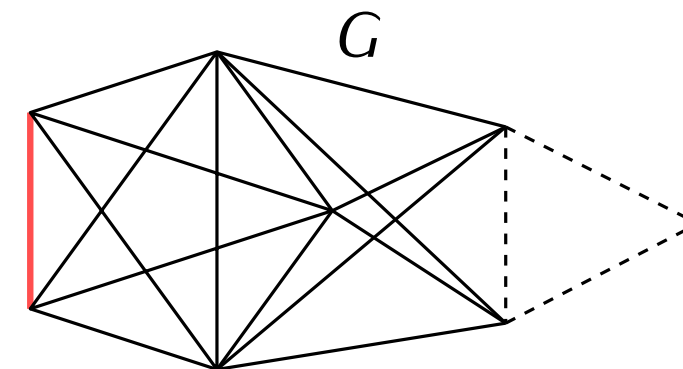


Instance of Not-All-Equal



Intuition:

- In the finite template, equality comes for free
- equality allows one to use "structural" constraints as "normal" constraints

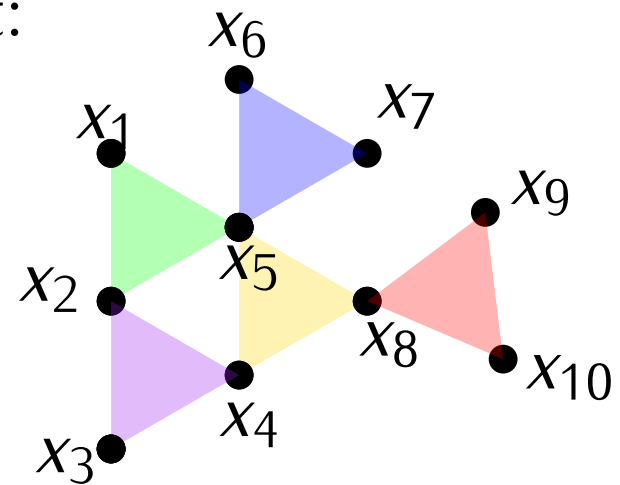


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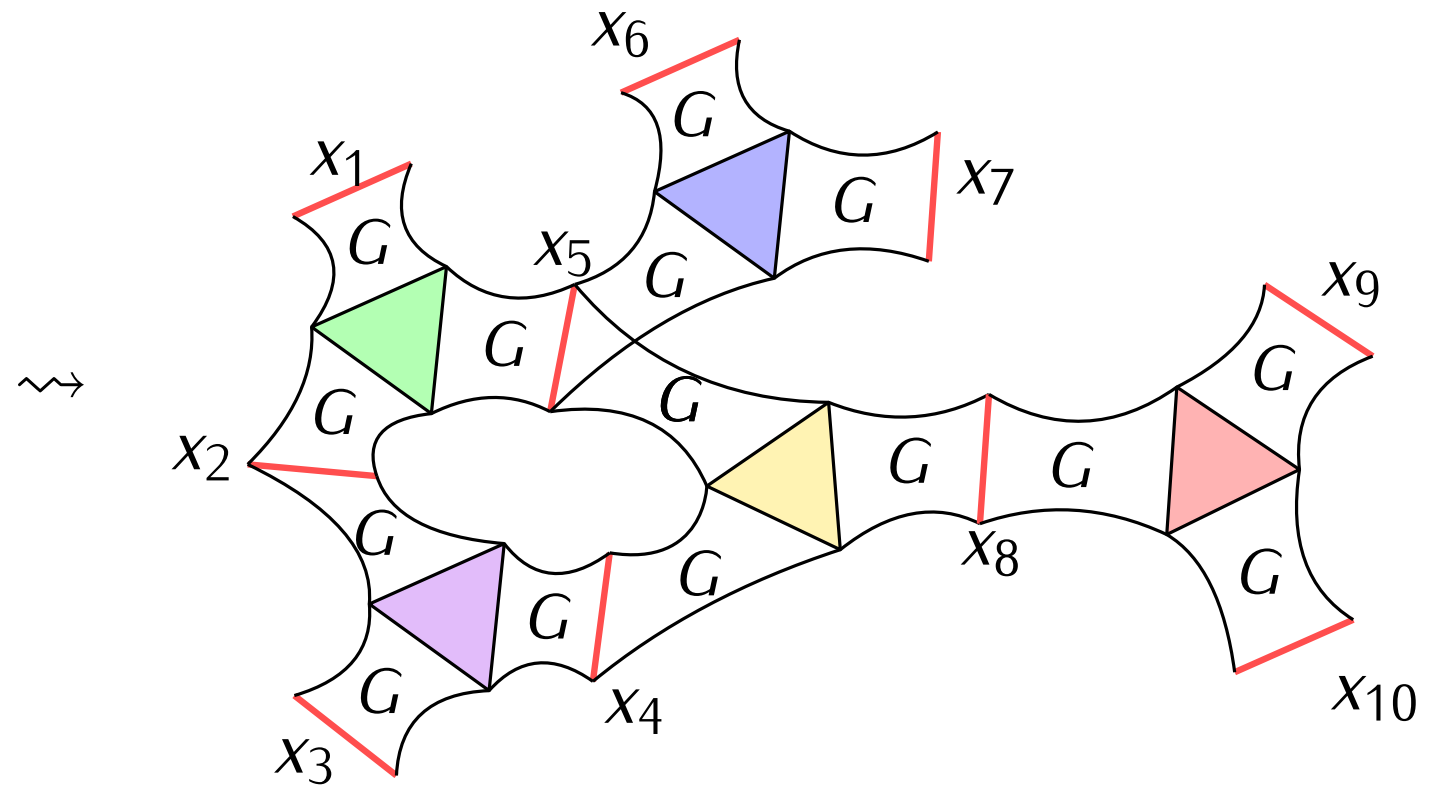
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Proposition (Barsukov, M., Perinti). Monochromatic Triangle remains hard on K_4 -free graphs.
(The gadget is beautiful but too large to fit on this slide)

