

Limitations of Affine Integer Relaxations for Solving Constraint Satisfaction Problems

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RWTHAACHEN
UNIVERSITY



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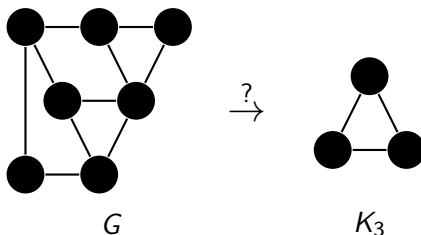
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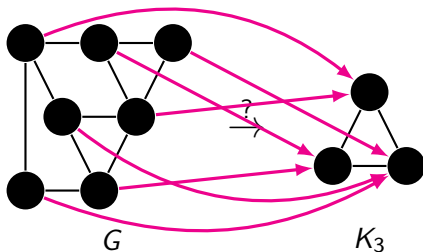


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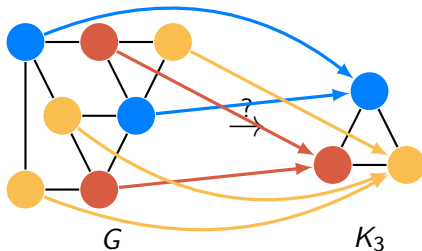


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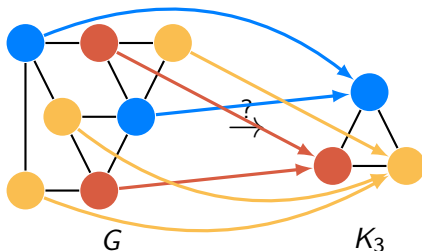


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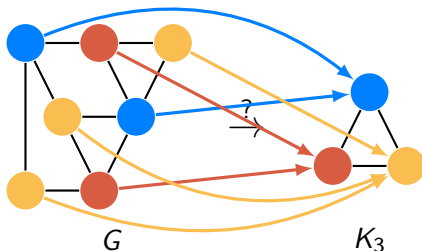
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- **dichotomy**: $\text{CSP}(A)$ is NP-complete or polynomial-time decidable (Bulatov, Zhuk)
- Is there a **universal algorithm** (polynomial in $|A|, |B|$) for polynomial-time CSPs?

Integer Linear System Formulation of CSPs

Encode existence of a global homomorphism $B \rightarrow A$.

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Equations for all $X \subseteq Y$ of size at most k and homomorphisms $f: B[X] \rightarrow A$:

$$\sum_{\text{hom } f: B[X] \rightarrow A} x_f = 1$$

$$x_f \in \{0, 1\}$$

is f used in global homomorphism?

X is uniquely mapped to A

$$\sum_{\substack{\text{hom } f: B[X] \rightarrow A \\ g|_Y = f}} x_g = x_f$$

f has a unique extension to Y

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- k -AIP: affine integral relaxation $x_f \in \mathbb{Z}$

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	3						
			1	9	5		
		8				6	
8				6			
4			8				1
				2			
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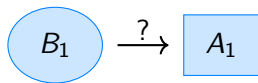
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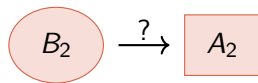
How to get a polynomial-time template?

Reduction to a Polynomial-Time Template

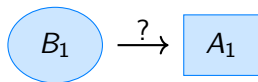
disjunction of
equation systems



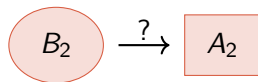
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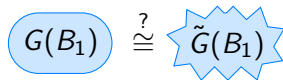
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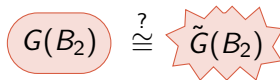
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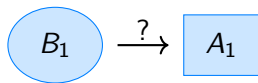
encoding as graph
isomorphism



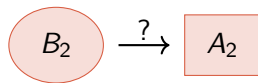
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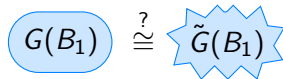
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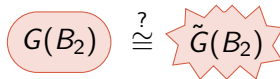
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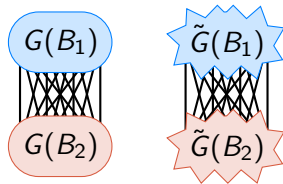
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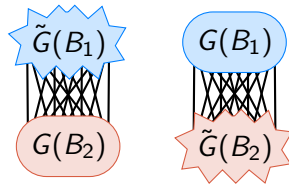
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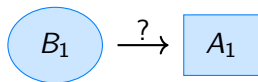
graph-isomorphism-or
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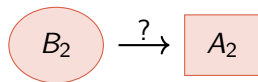
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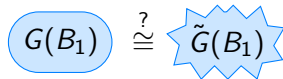
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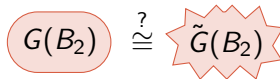
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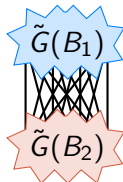
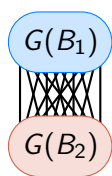
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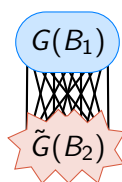
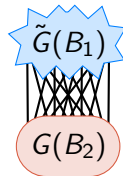
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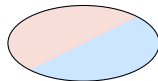
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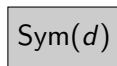
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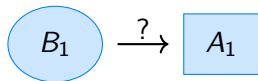
encoding as
group-coset CSP



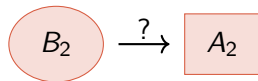
$\xrightarrow{?}$



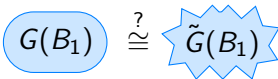
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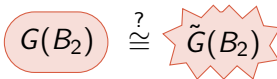
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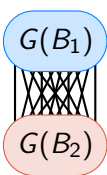
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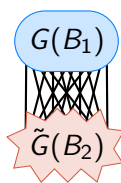
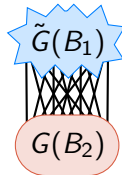
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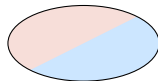
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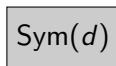
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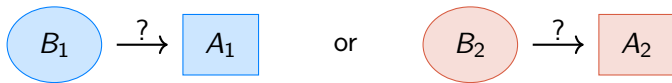
Theorem (L., Pago)

For all k , the \mathbb{Z} -affine k -consistency algorithm does not solve $\text{CSP}(\text{Sym}(18))$.

Refuting the Conjecture: Homomorphism-Or Construction

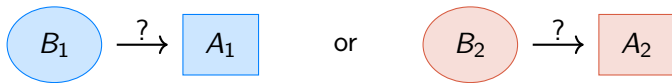
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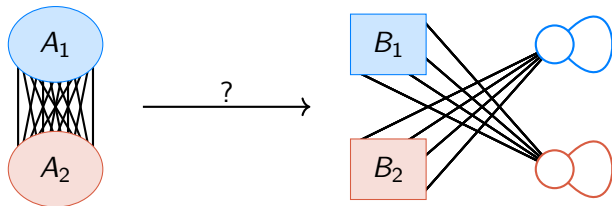


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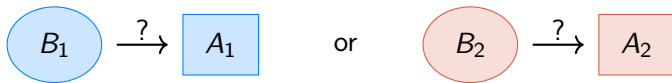


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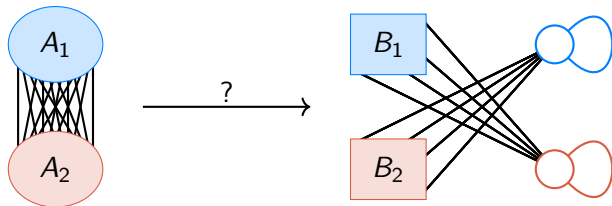


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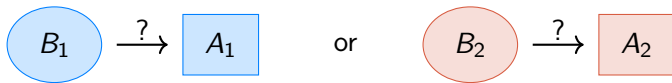


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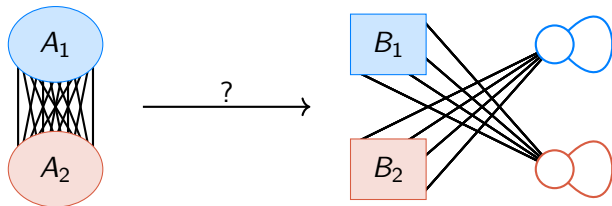
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 k -consistency

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There are more CSP-algorithms combining combinatorial approaches with solving \mathbb{Z} -linear systems:

\mathbb{Z} -affine
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BLP+AIP

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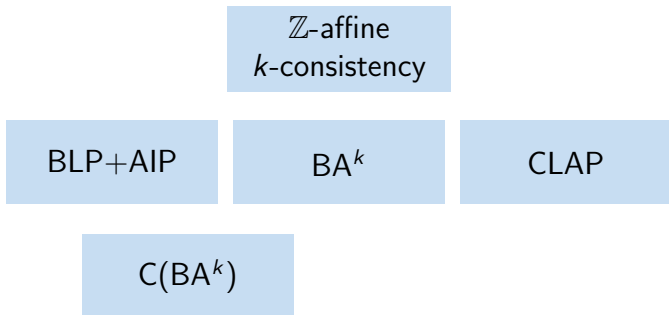
BLP+AIP

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CLAP

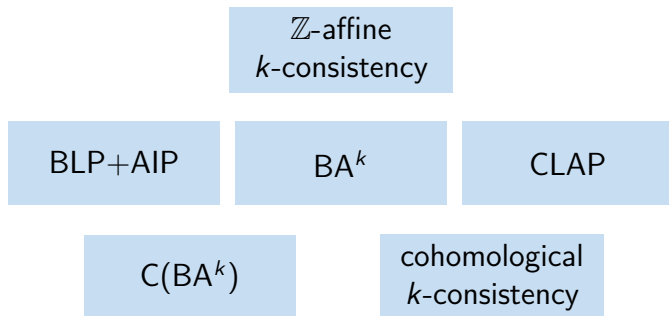
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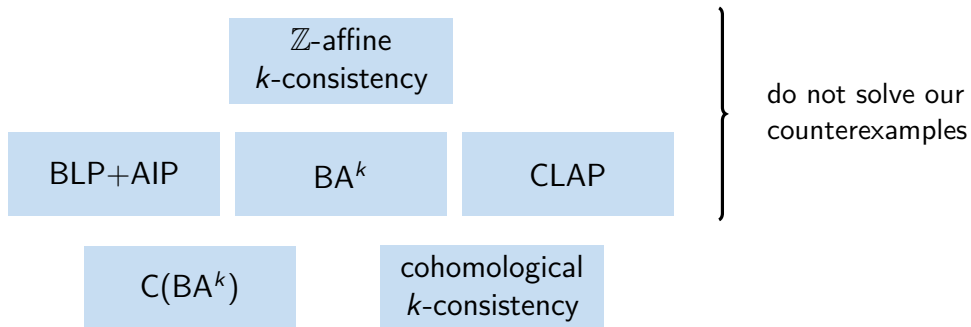
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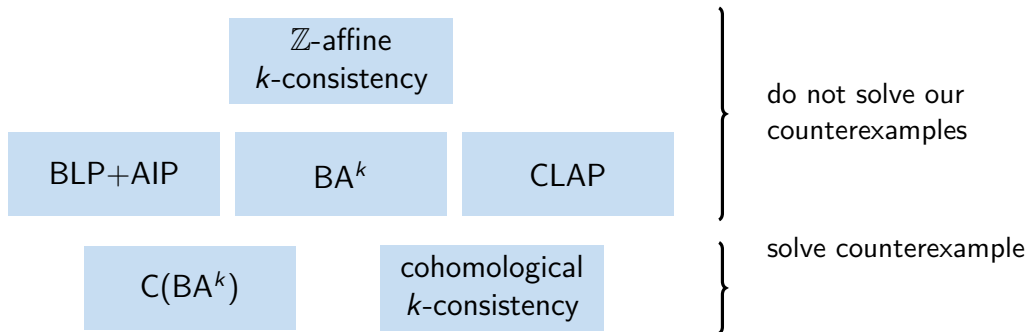
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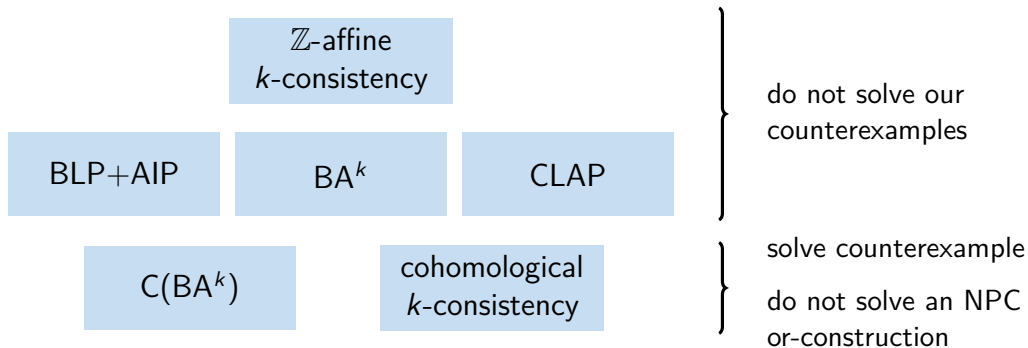
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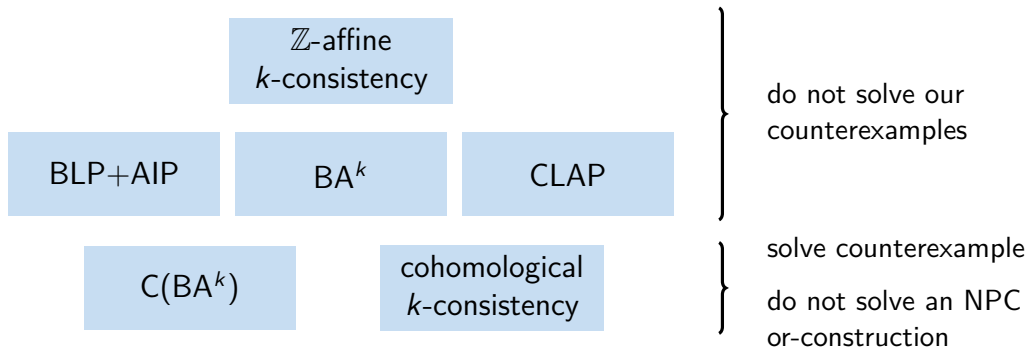
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Open Problem

- polynomial-time counterexample for cohomological k -consistency?