

# Separability properties of monadically dependent graph classes

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FO model checking over  $\mathcal{C}$ :

Given: A FO-sentence  $\phi$  and a graph  $G \in \mathcal{C}$ .

Parameter:  $|\phi|$ .

Decide:  $G \models \phi$ .

## Conjecture

*A hereditary class admits FPT first-order model checking iff it is monadically dependent.*

where: monadic dependence = does **not** transduce the class of all graphs

The conjecture has been confirmed for:

- $K_{t,t}$ -free classes (where mon. dependence = nowhere density, [GKS17])
- edge-stable classes (where mon. dependence = mon. stability, [DEM<sup>+</sup>24])
- ordered classes (where mon. dependence = bdd twin-width, [BGOdM<sup>+</sup>22])

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Theorem (Dreier, Mählmann, Toruńczyk, 2024)

*Model checking on hereditary independent classes is AW[\*]-hard.*

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...we have insufficient combinatorial understanding of monadic dependence.

# Characterisations of nowhere density

Originally defined in terms of the exclusion of shallow minors, nowhere dense classes have been shown to admit various equivalent characterisations:

- forbidden subgraphs
- quasi-bounded bounded treedepth covers
- Splitter game
- sparse neighbourhood covers
- neighbourhood complexity
- uniform quasi-wideness
- “local separability”
- ...

## Lemma

*For every  $G$  and  $\varepsilon > 0$  there exists  $k := k(\text{tw}(G), \varepsilon)$  such that every connected component of  $G$  contains fewer than  $\varepsilon \cdot |G|$  vertices after deleting at most  $k$  vertices.*



# Separability in sparse classes

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## Theorem (Nesetril, Ossona de Mendez, 2016)

*Let  $\mathcal{C}$  be nowhere dense. Then for every  $r \in \mathbb{N}$  and  $\varepsilon > 0$  there exists  $k := k(\mathcal{C}, r, \varepsilon)$  such that for every  $G \in \mathcal{C}$  there is some  $S \subseteq V(G)$  with  $|S| \leq k$  satisfying:*

$$\frac{|\text{Ball}_{G \setminus S}^r(v)|}{|G|} < \varepsilon, \quad \text{for all } v \in V(G)$$

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## Theorem (Nesetril, Ossona de Mendez, 2016)

A class  $\mathcal{C}$  is nowhere dense *if and only if* for every  $r \in \mathbb{N}$  and  $\varepsilon > 0$  there exists  $k := k(\mathcal{C}, r, \varepsilon)$  such that for every  $G \in \mathcal{C}$  and *every*  $A \subseteq V(G)$  there is some  $S \subseteq A$  with  $|S| \leq k$  satisfying:

$$\frac{|\text{Ball}_{G[A \setminus S]}^r(v)|}{|A|} < \varepsilon, \quad \text{for all } v \in V(G)$$

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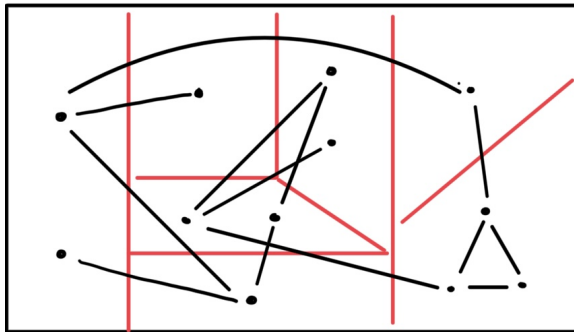
*For every  $G$  and  $\varepsilon > 0$  there exists  $k := k(\text{cw}(G), \varepsilon)$  and a  $k$ -flip  $G'$  of  $G$  such that every connected component of  $G'$  contains fewer than  $\varepsilon \cdot |G|$  vertices.*



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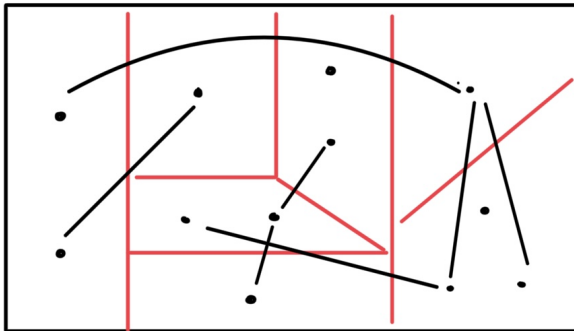
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## Theorem (BBEGMPPT, 2025)

A class  $\mathcal{C}$  is monadically dependent if and only if for every  $r \in \mathbb{N}$  and  $\varepsilon > 0$  there exists  $k := k(\mathcal{C}, r, \varepsilon)$  such that for every  $G \in \mathcal{C}$  and every  $A \subseteq V(G)$  there is some  $k$ -flip  $G'$  of  $G$  satisfying:

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$$\mu(\text{Ball}_{G'}^r(v)) < \varepsilon, \quad \text{for all } v \in V(G)$$

We call this property *flip-separability*.

Flip-separability  $\implies$  monadic dependence

### Lemma

*Flip-separability is closed under transductions.*

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Proof: For every transduction  $\mathcal{T}$  involving  $c$  colours and formulas of quantifier rank at most  $q$ , any two graphs  $G, H$  such that  $H \in \mathcal{T}(G)$ , and any  $k$ -flip  $G'$  of  $G$  there exists a  $\Xi(k, c, q)$ -flip  $H'$  of  $H$  such that for all  $u \in V(H)$ :

$$\text{Ball}_{H'}^r(u) \subseteq \text{Ball}_{G'}^{2^q \cdot r}(u)$$

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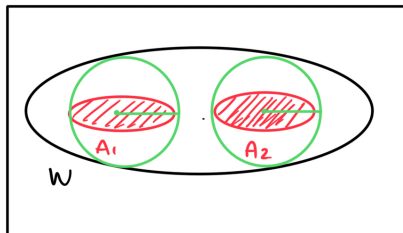
Proof: Any flip-separable class has bounded VC-dimension.

## Theorem (Dreier, Möhlmann, Toruńczyk, 2024)

*Every monadically dependent graph class  $\mathcal{C}$  is flip-breakable, i.e.*

*for every  $r \in \mathbb{N}$  there exists  $t := t(\mathcal{C}, r)$  and a function  $M_r : \mathbb{N} \rightarrow \mathbb{N}$  such that for every graph  $G \in \mathcal{C}$ ,  $m \in \mathbb{N}$ , and set  $W \subseteq V(G)$  of size at least  $M_r(m)$ , there are disjoint subsets  $A_1 \subseteq W, A_2 \subseteq W$  each of size at least  $m$  and a  $t$ -definable flip  $H$  of  $G$  satisfying*

$$\text{Ball}_H^r(A_1) \cap \text{Ball}_H^r(A_2) = \emptyset$$



# Monadic dependence $\implies$ flip-separability

Proof idea:

- Fix  $r \in \mathbb{N}$  and  $\varepsilon > 0$ . Let  $G \in \mathcal{C}$  and  $S \subseteq V(G)$  be of minimal size such that there is  $S$ -definable flip of  $G$  witnessing that all the  $r$ -balls are  $\varepsilon$ -small. We know that  $S = V(G)$  works;

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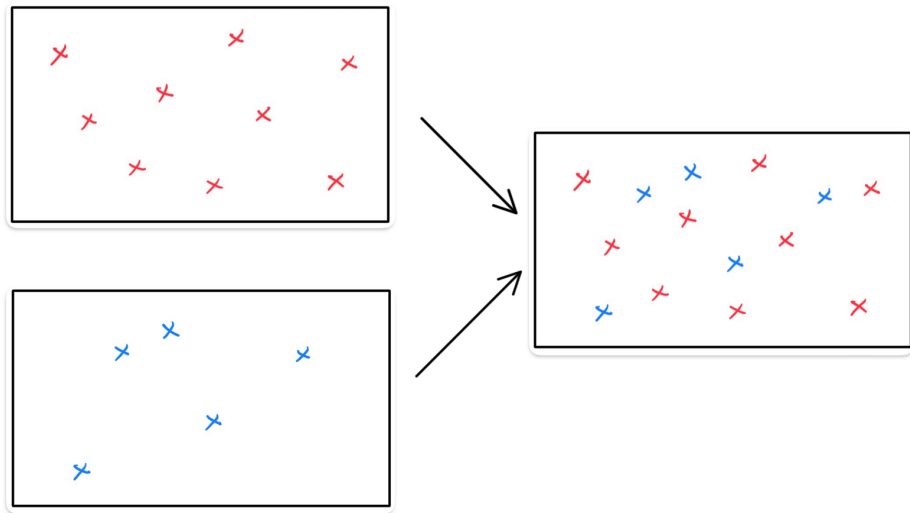
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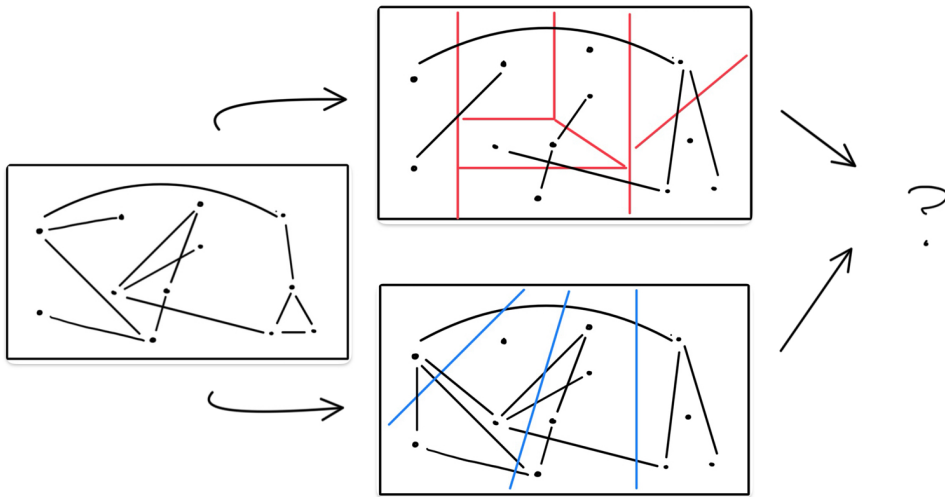
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- In this  $F$ -flip, one of the  $A_i$ 's must have large measure
- Ideally we would want to remove  $A_i$  from  $S$ , but we should also add  $F$  back to it...

We want to somehow combine the information coming from multiple flips.

# Aggregation of deletions



# Aggregation of flips..?



# Partition metric

For any  $\mathcal{P}$  partition of  $V(G)$  we define

$$\text{dist}_{\mathcal{P}}^G(u, v) := \max \{ \text{dist}_{G'}(u, v) : G' \text{ is a } \mathcal{P}\text{-flip of } G \},$$

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Intuition: things are far apart in  $\text{dist}_{\mathcal{P}}$  iff this is witnessed by some  $\mathcal{P}$ -flip



# Metric conversion

Sometimes we only care about distance, and sometimes we care about concrete structure.

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It turns out that these perspectives are interchangeable:

## Theorem

*For every graph  $G$  of VC-dimension  $d$  and a partition  $\mathcal{P}$  of  $V(G)$ , there is a set of vertices  $S$  of size  $\mathcal{O}(d \cdot |\mathcal{P}|^{2d+2})$  and an  $S$ -definable flip  $G'$  of  $G$  such that for every  $r \in \mathbb{N}$ ,  $v \in V(G)$*

$$\text{Ball}_{G'}^r(v) \subseteq \text{Ball}_{\mathcal{P}}^{30r}(v).$$

# Monadic dependence $\implies$ flip-separability

Proof idea:

- Fix  $r \in \mathbb{N}$  and  $\varepsilon > 0$ . Let  $G \in \mathcal{C}$  and  $S \subseteq V(G)$  be of minimal size such that all the  $r$ -balls are  $\varepsilon$ -small **in the  $S$ -metric**. We know that  $S = V(G)$  works;

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- In this  $F$ -flip, one of the  $A_i$ 's must have large measure;
- Take  $S' = (S \setminus A_i) \cup F$ .

This does the trick, but it gets slightly technical...

Thank you!





Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk.

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