Representations of permutation groups in Fixed-point logic

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Motivation
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Plan



- 2 First approach: definable generating sets
- 3 Morphism-definability
- 4 Ordered sets of permutations

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Quest of a logic for P, short recap

- Find an enumeration of the P queries (more or less: [Gur88])
- Immerman-Vardi theorem: FP over ordered structures
- Over arbitrary structures:
 - FP does not express Parity

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 - FP + rk ≠ P [Lic23]
 - CPT ?

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 - FP + C captures P on any class of structures with bounded tree-width [GM99]
 - More generally, FP + C captures P on any class of structures which excludes a minor [Gro17].

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 - Relies on canonisation

The role of permutation groups in Graph Isomorphism

- Graph Automorphism problem (GA_C): given G ∈ C, output a generating set for Aut(G) ≤ Sym(V_G).
- Note: a polynomial-size such generating set always exists.
- $GI_{\mathcal{C}} \leq_{P} GA_{\mathcal{C}}$ for any union-closed class of (connected) graphs \mathcal{C} .
- Based on this insight, many upper bounds have been found for Graph Iso and/or Canonisation for restricted (or not) classes of graphs [Bab79, Luk82, BL83, Bab16]. Most use the CFSG. One exception: Bounded Colour-class Graph Isomorphism.

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Primitive operations on permutation groups

All those results rely on some fundamental operations on permutation groups which can be carried out in polynomial time, thanks to the Schreier-Sims algorithm:

- Given $S \subseteq \text{Sym}(D)$, output $|\langle S \rangle|$.
- Given $S \subseteq \operatorname{Sym}(D)$ and $\sigma \in \operatorname{Sym}(D)$, does $\sigma \in \langle S \rangle$?
- Given S ⊆ Sym(D) and a black-box membership test for H ≤ ⟨S⟩ such that |⟨S⟩|/|H| < |D|^k (for some fixed k), output T ⊆ Sym(D) s.t. ⟨T⟩ = H.

This last operation gives rise to the definition of *k*-accessible subgroups of $\langle S \rangle$.

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How to represent permutation groups in relational structures ?

Can we bring those methods in reach of isomorphism-invariant formalisms for polynomial-time computation ?

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How to represent permutation groups in relational structures ?

- Can we bring those methods in reach of isomorphism-invariant formalisms for polynomial-time computation ?
- One main issue: it is not clear how to represent permutation groups in relational structures.

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1 Motivation

2 First approach: definable generating sets

3 Morphism-definability

4 Ordered sets of permutations

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Direct translation of the algorithmic framework

Definition (definable permutation)

A permutation $\sigma \in \text{Sym}(A^k)$ is definable in \mathfrak{A} if there is a formula $\varphi(\vec{s}, \vec{t})$ with $|\vec{s}| = |\vec{t}| = k$ such that

$$\forall \vec{b}, \vec{c} \in A^k, \sigma(\vec{b}) = \vec{c} \iff (\mathfrak{A}, \vec{b}, \vec{c}) \models \varphi$$

Definition (definable permutation group)

A group $G \leq \text{Sym}(A^k)$ is definable in \mathfrak{A} if there is a formula $\varphi(\vec{p}, \vec{s}, \vec{t})$ such that

$$\langle \{\operatorname{perm}(\varphi(\mathfrak{A}, \vec{a})) \mid \vec{a} \in A^{\vec{p}} \} \rangle = G$$

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Low-hanging fruits

- FO + tc can define the orbits of any definable group.
- (Schreier-Sims) Given any structure \mathfrak{A} and any formula φ (in a poly-time model checking logic), $|\langle \varphi \rangle|$ is computable in polynomial time.
- If G and H are \mathcal{L} -definable, with $\mathcal{L} \geq \text{FO}$, $\langle G \cup H \rangle$ is \mathcal{L} -definable.

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- Corollary: the membership problem reduces (via "Turing FO reduction") to group-order computation.

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- Turing FO reduction: What if we add an operator to FP that computes the order of a group ? ord

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Spoiler

Theorem (D.25)

FP + rk < FP + ord.

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Limits

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In an isomorphism-invariant context, this representation is not *complete*, i.e. there are groups which admit no small, isomorphism-invariant, generating set:

$$\mathfrak{A}_n := K_{n,n}$$
 and $G_n := \operatorname{Aut}(\mathfrak{A}_n)$

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Worse, we can find a (sequence of) structure \mathfrak{A} s.t.:

- A group $\mathbf{G}(\mathfrak{A})$ is (FP-)definable from \mathfrak{A}
- A subgroup $H(\mathfrak{A})$ is accessible from $G(\mathfrak{A})$
- FP defines a witness of the accessibility of $H(\mathfrak{A})$ in $G(\mathfrak{A})$.
- Any symmetric generating set of $H(\mathfrak{A})$ has exponential size.

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Idea: find restricted cases where we can leverage structural properties of the groups at hand to represent them differently.

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Plan

1 Motivation

2 First approach: definable generating sets

3 Morphism-definability

4 Ordered sets of permutations

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Morphism-definability

Definition

Let *R* be a relation symbol of arity 2k. $\varphi_m(R, \vec{x}, \vec{y})$ defines a morphism $m : G \to \text{Sym}(A^{|\vec{x}|})$ on \mathfrak{A} , where $G \leq \text{Sym}(A^k)$ if, for all $\sigma \in G$,

$$\varphi_m(\mathfrak{A}, R \leftarrow \operatorname{graph}(\sigma)) = \operatorname{graph}(m(\sigma))$$

 $H \trianglelefteq G \le \operatorname{Sym}(A^{\mathcal{T}})$ is morphism-definable from G in \mathfrak{A} if:

- There is a definable generating set for G in \mathfrak{A}
- there is a \mathcal{L} -formula φ_m which defines a morphism $m: G \to \operatorname{Sym}(A^{T'})$ on \mathfrak{A} such that ker(m) = H.

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Operations on morphism-definable subgroups

- If *H* is morphism-definable from *G* in 𝔅, then FP + ord defines |*H*|, and defines membership to *H*.
- If H_1 and H_2 are morphism-definable from G in \mathfrak{A} , then so is $H_1 \cap H_2$.



Implications

- The separation of FP + rk from P relies on structures with abelian colours. In this case, all the relevant groups are morphism-definable.
- Yields a definable canonisation of those structures (following the algorithm from [BL83])
- Theorem: FP + rk < FP + ord [D.25].

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Plan

1 Motivation

2 First approach: definable generating sets

3 Morphism-definability

4 Ordered sets of permutations

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- If FP + ord defines an ordered generating set of permutations for G on 𝔄, and H ≤ G is FP + ord-definably accessible in G, then FP + ord defines a generating set for H.
- if *G* is abelian, FPC suffices.
- structures with abelian colours are equipped with ordered, abelian groups.
- FPC defines the automorphism groups of the structures with abelian colours (which separate FP + rk from P).

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Conclusion

Thank you !

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