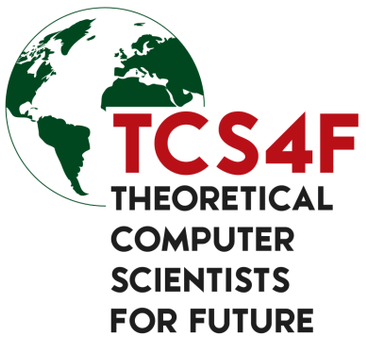
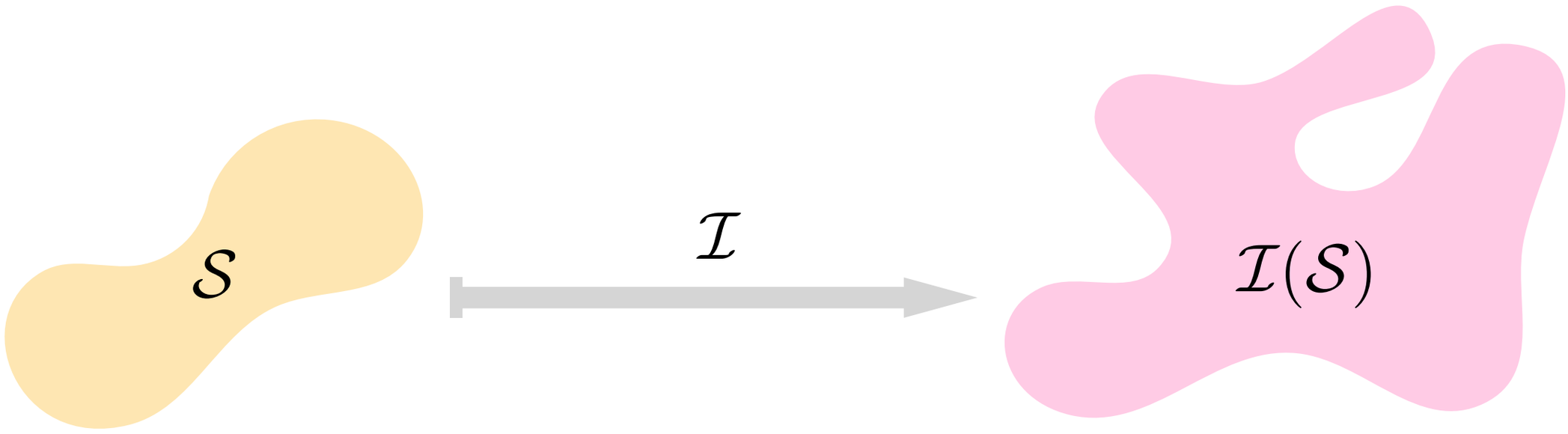
String-to-string MSO-set-interpretations

Thomas Colcombet, 30 May 2025, Finite Model Theory, Les Houches

joint work with Nathan Lhote, Lê Thành Dũng (Tito) Nguyễn and Pierre Ohlmann.



FO-interpretation of dimension k



Input structure of universe U.

Output structure
Elements are k-tuples of element of S.
Relations defined in S by FO-formulas.

Eg. Path $\longmapsto$ grids.

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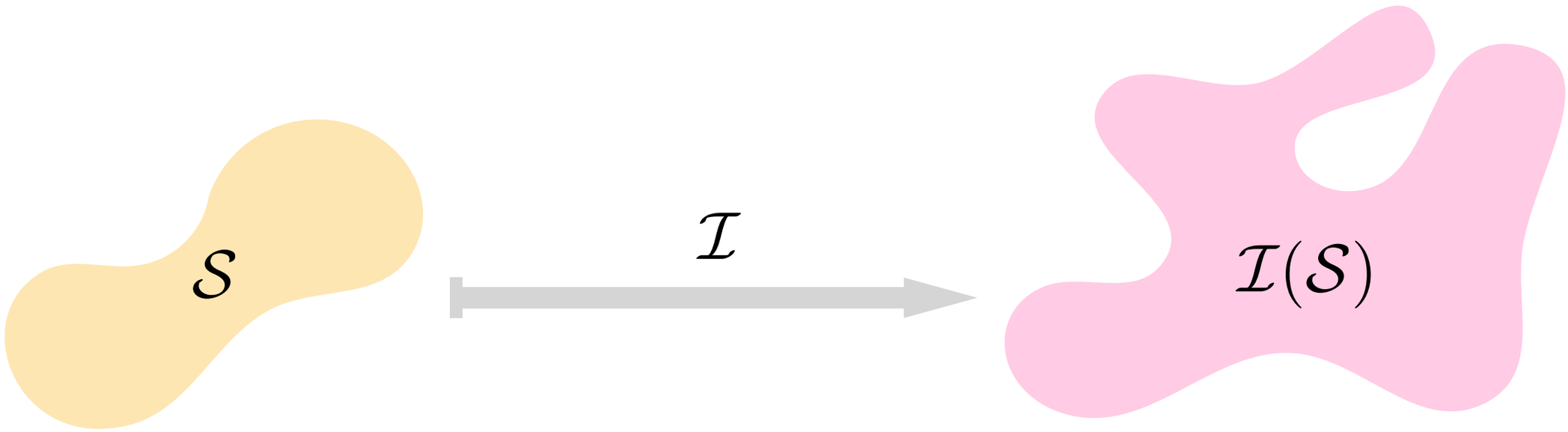

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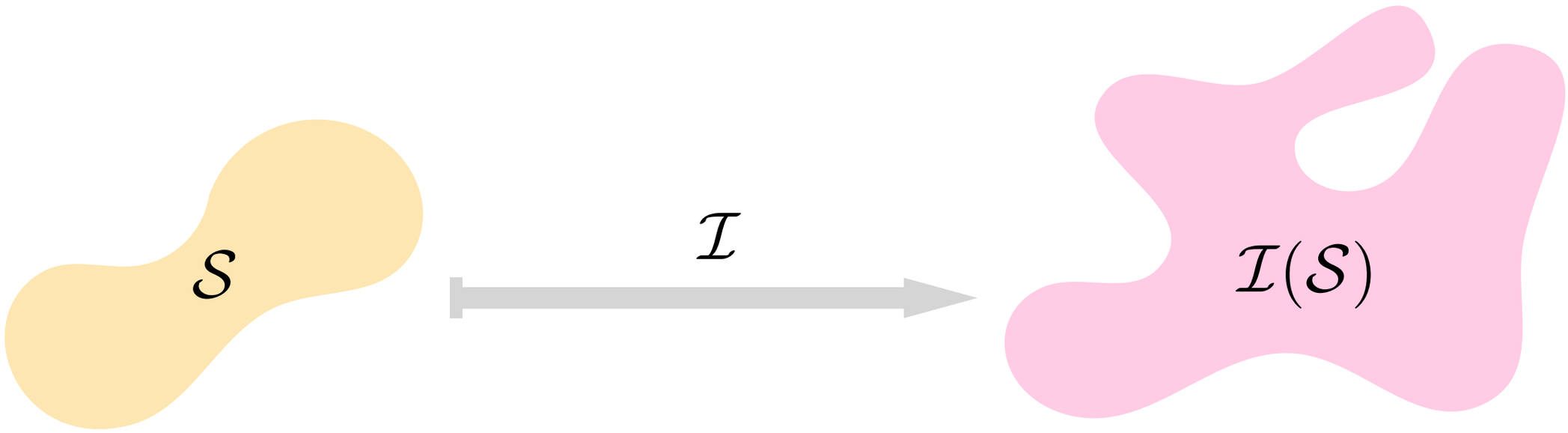
(parameterless) MSO-transductions
= MSO-interpretation 1-dimensional with k-copying



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Universe MSO-definable in U $\times$ [k] .
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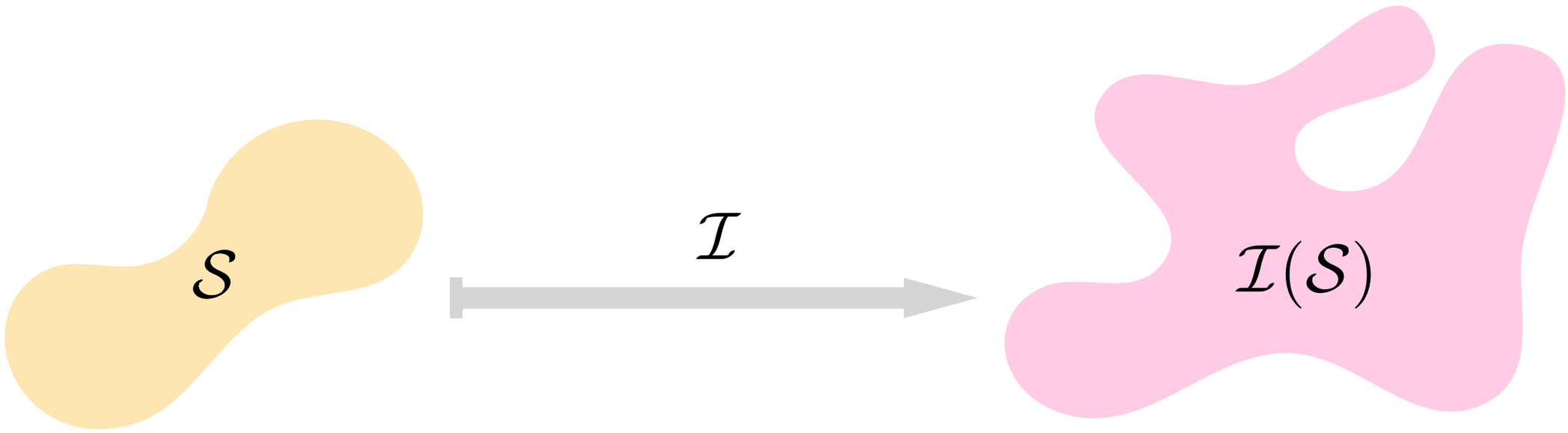

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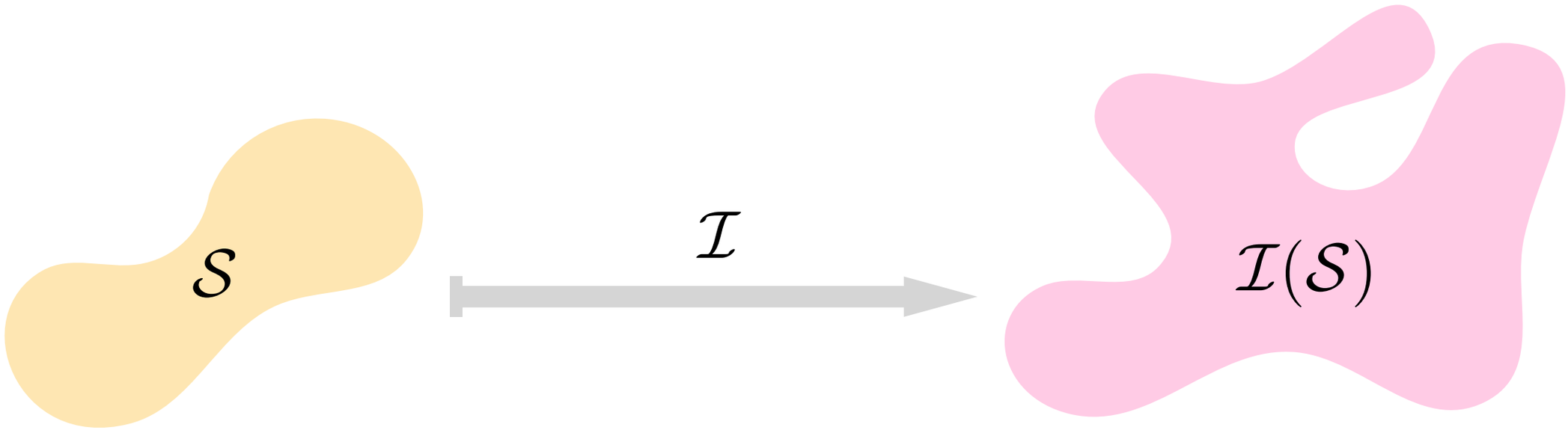

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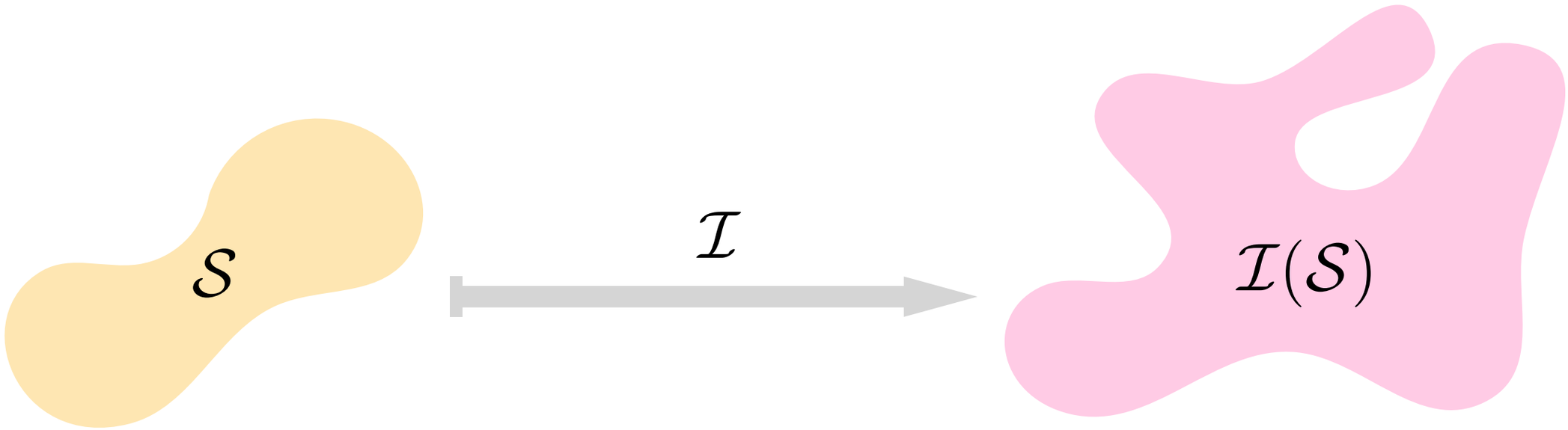

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For MSO-transductions = MSO-interpretations of dimension 1	$\mathcal{U} \times k$	The inverse image of MSO-definable is effectively FO-definable
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Theorem [Bojańczyk, Kiefer, Lhote19]:

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Finite linear orders
+ unary predicates

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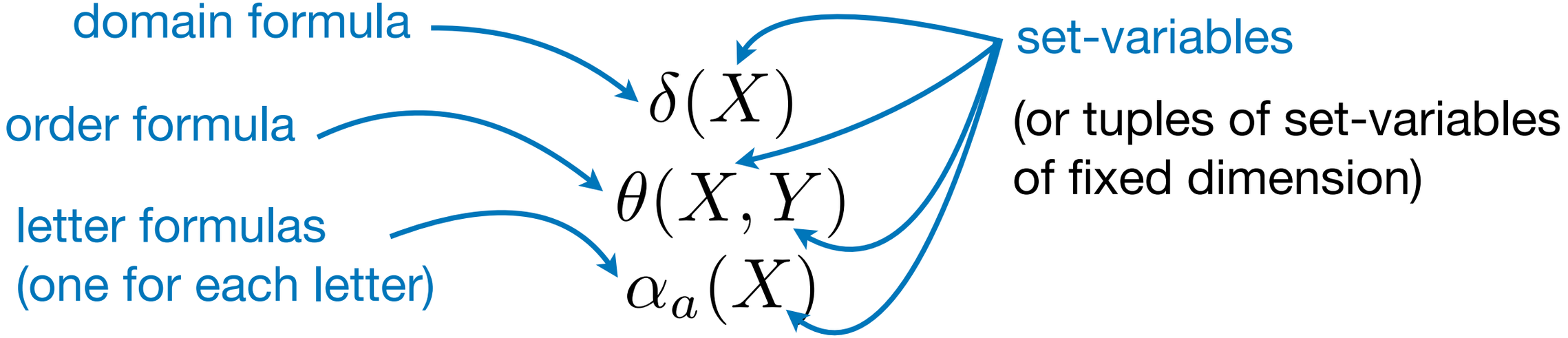
String-to-string MSO-set-interpretation

Input and output signature: linear order symbol + unary predicates (letters)

A **string-to-string MSO-set-interpretation** consists of a tuple of MSO-formulas over the input signature:



Given an **input string**, it defines the **output string** consisting of:

- positions are subsets of the input that satisfy the domain formula
- the order is interpreted using the order formula (it is required to produce a total order!)
- the unary predicates (letters) are interpreted using the letter formulas.

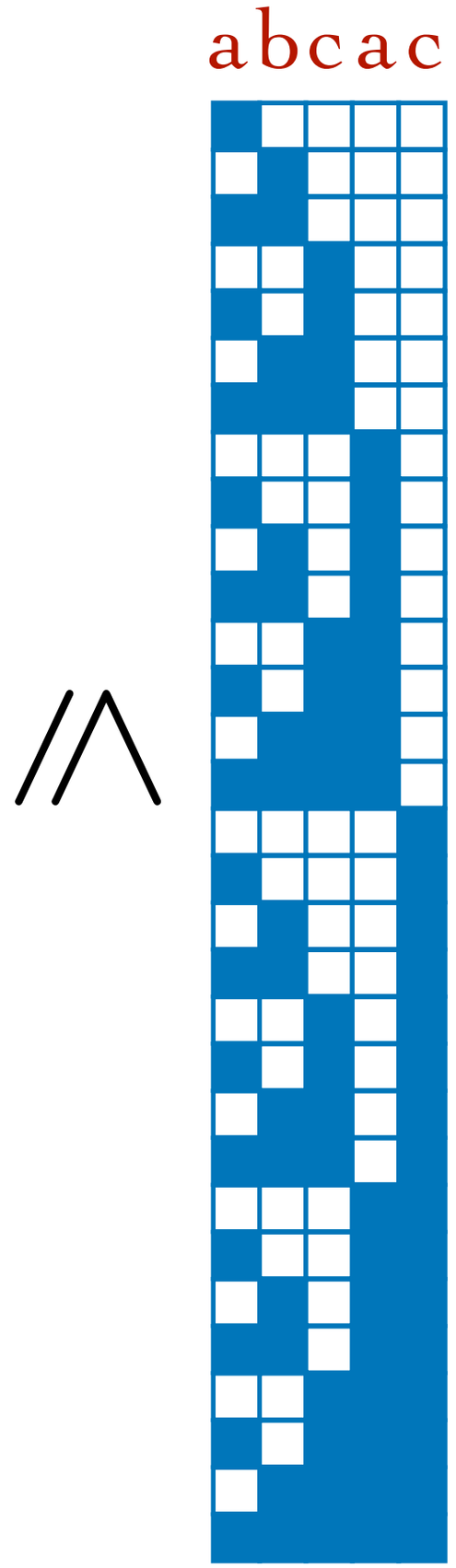
String-to-string MSO-set-interpretation

Example:

domain = non-empty sets

order = lexicographic

letter = the input letter at the maximal position in the set



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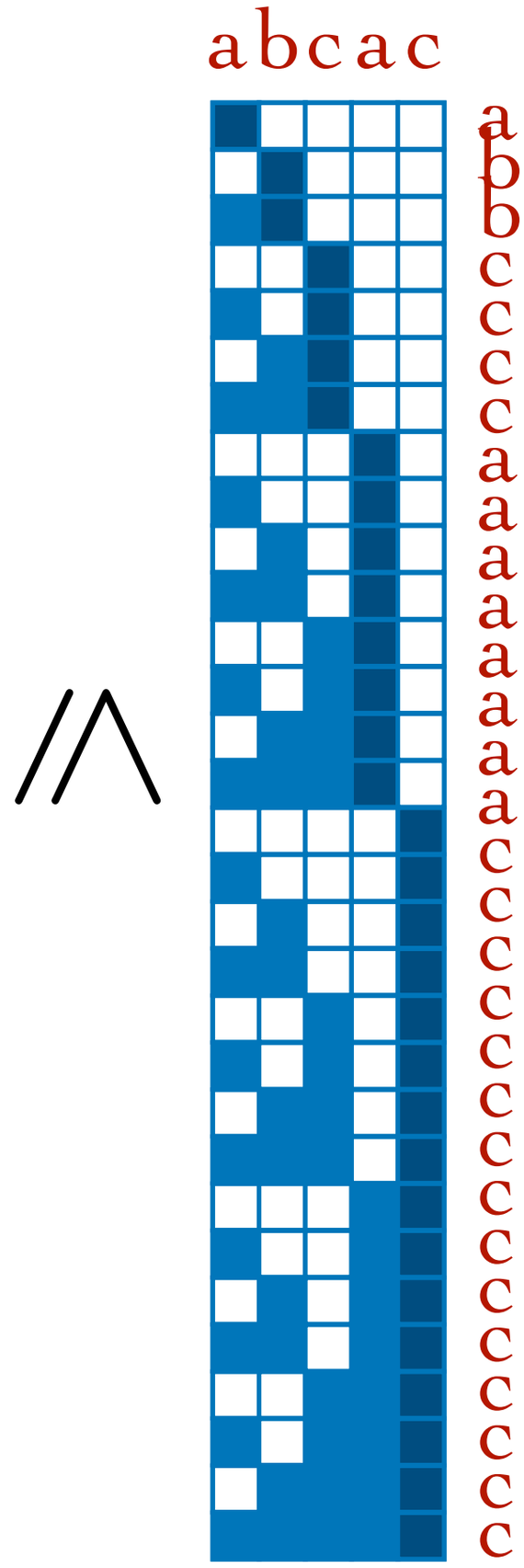

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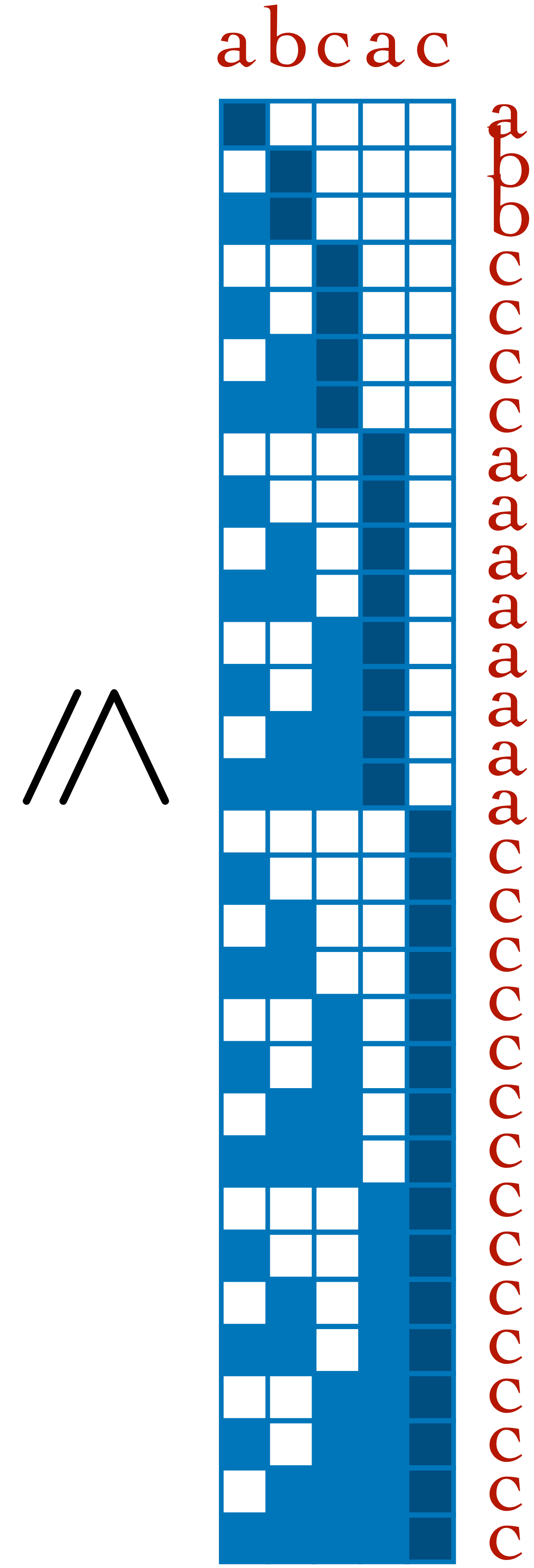

$$f: \quad \Sigma^* \to \Sigma^*$$

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Polyregular functions correspond to the special case where tuples of first-order variables are used instead of tuples of sets (still MSO formulas) = MSO-interpretation of dimension k.

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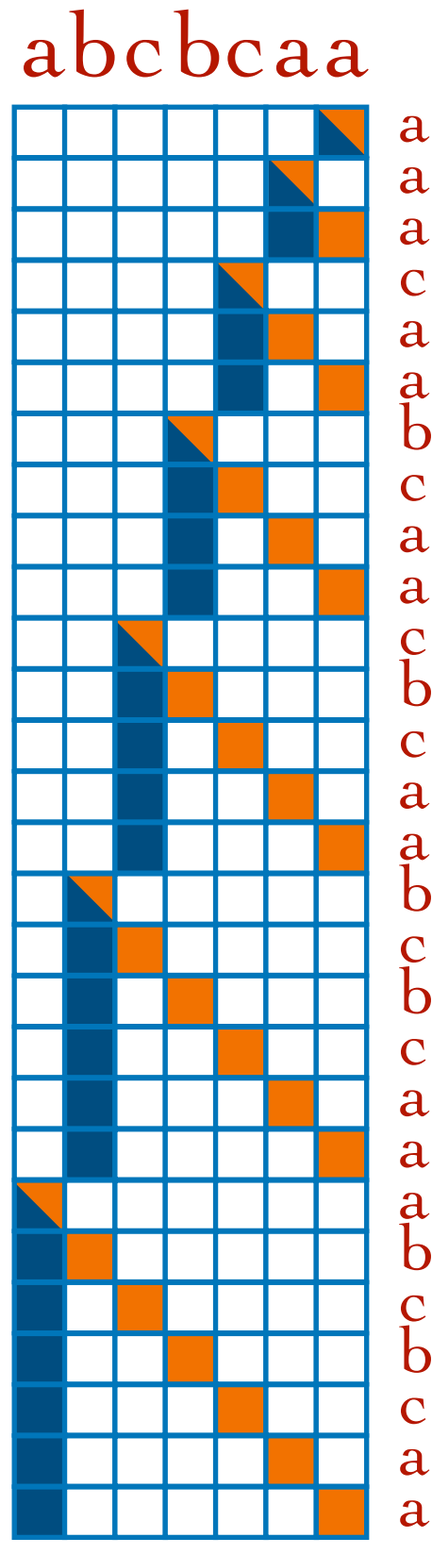

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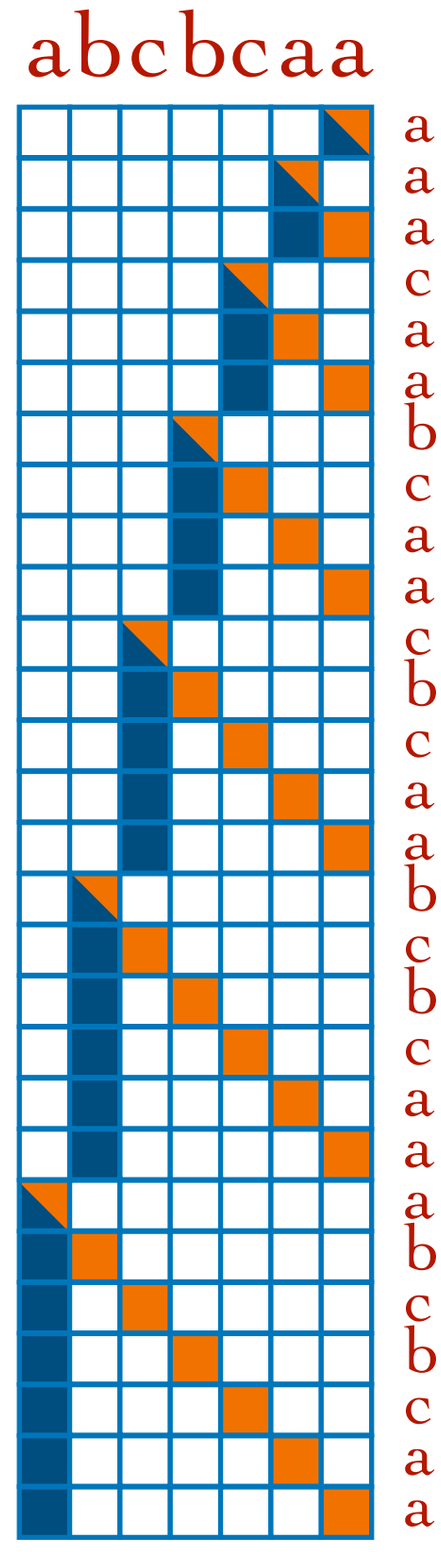

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Main result

Theorem: For a string-to-string map, the following items are effectively equivalent:

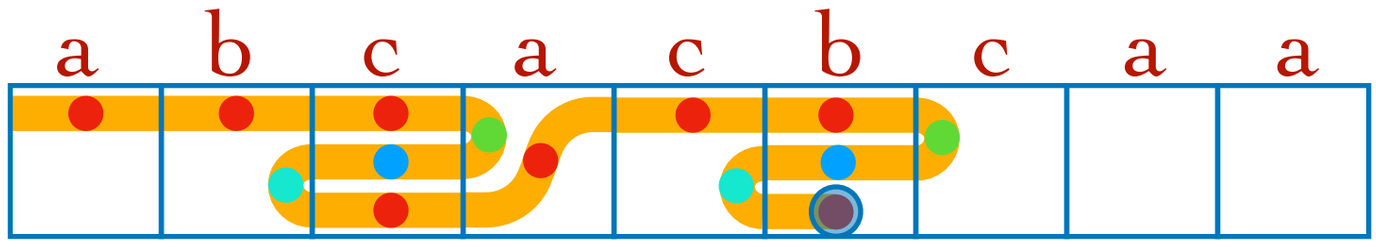
- being definable by an MSO-set-interpretation,
- being definable by an Ariadne-transducer,
- being definable by an Ariadne-transducer with regular lookaround of the configuration,
- being definable by a push-only string-to-tree Ariadne-transducer followed by a yield.

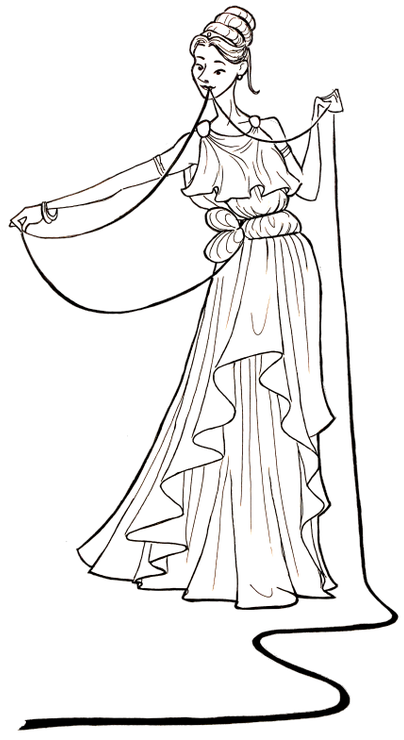
Furthermore, the inverse image of a regular string language under such a map is effectively regular.

Joséphine C.

String-to-string Ariadne-transducer

A **configuration** of an **Ariadne-transducer** is a walk on the input word, decorated with **states** (from a finite set), that can visit at most k times each position of the input string:





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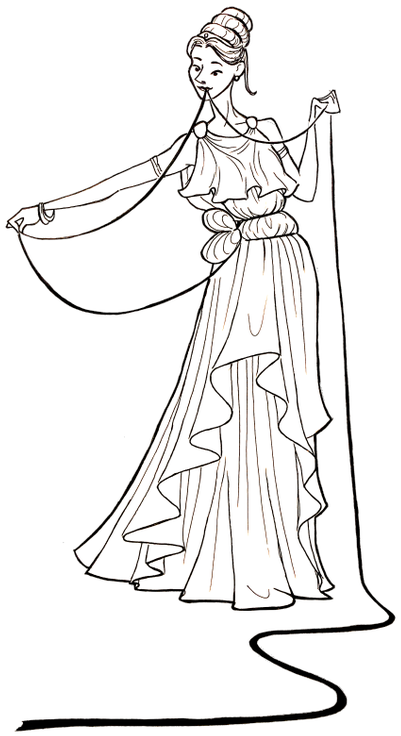

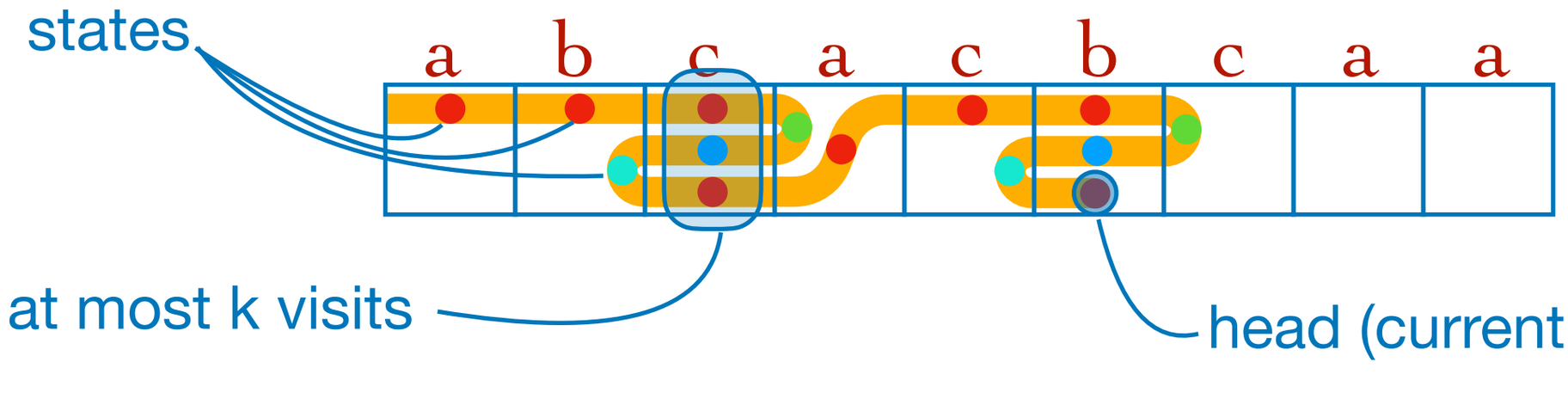

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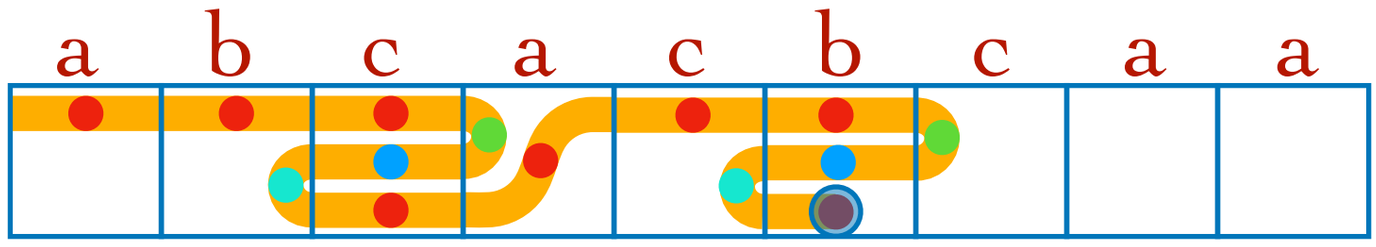

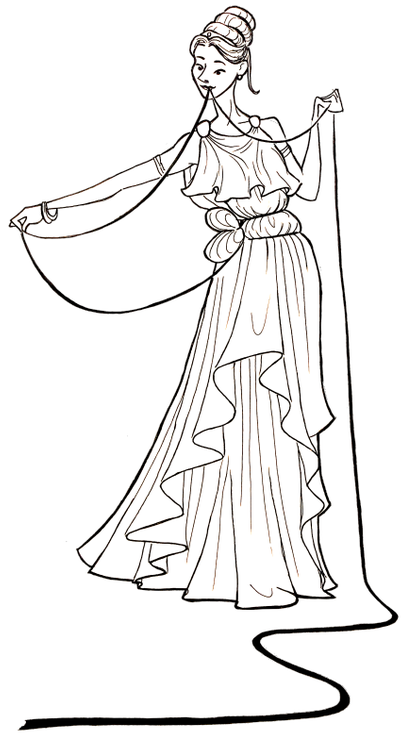

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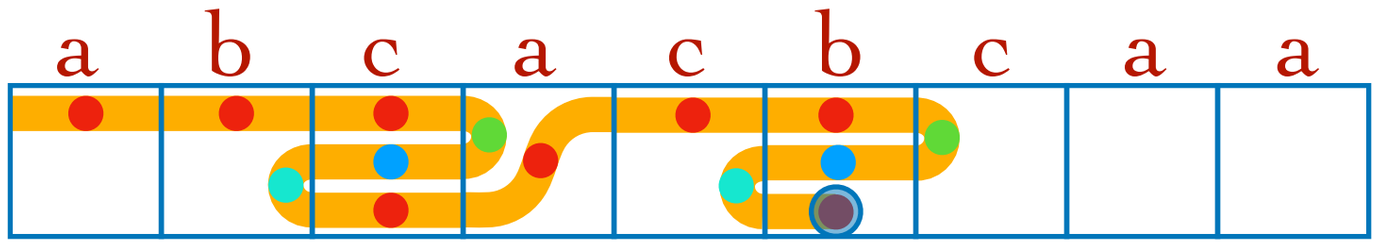

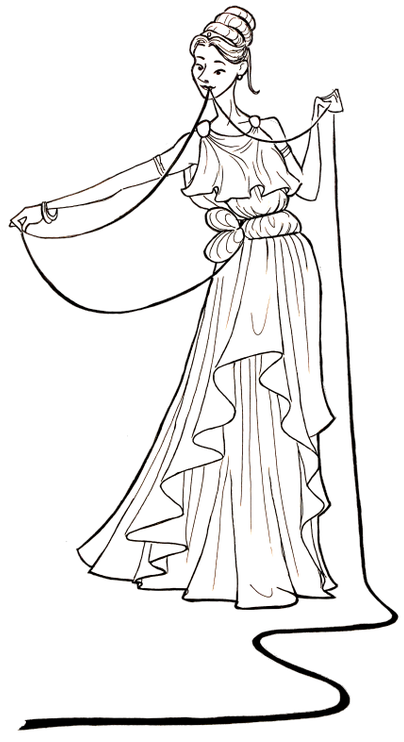

A **transition** is decided based on the full content of the configuration at the head position (input letter + states), and can possibly either

- **[push]** extend the walk of one step (left or right), or
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and at the same time it

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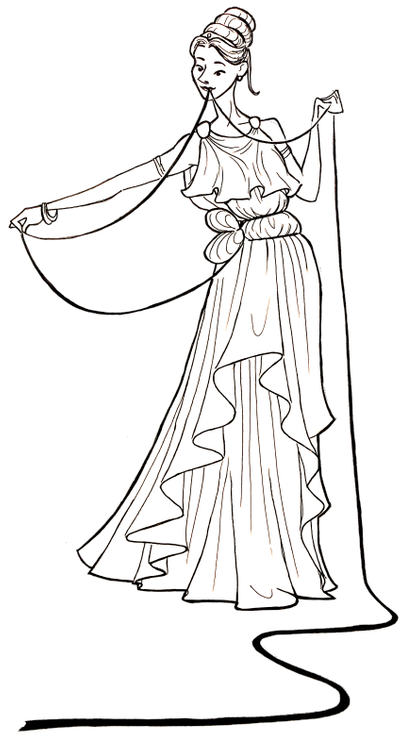

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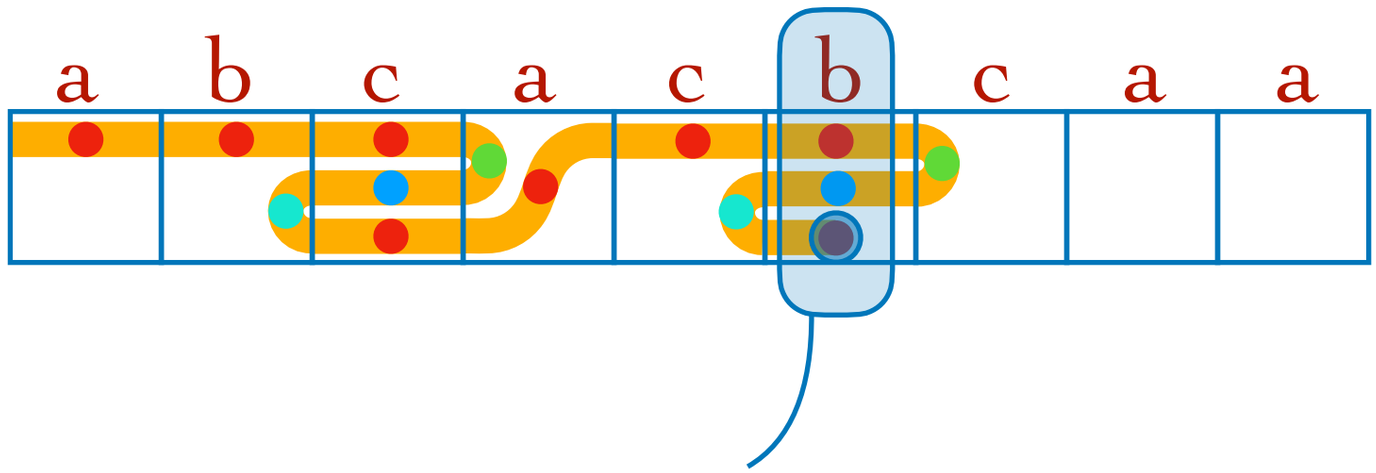

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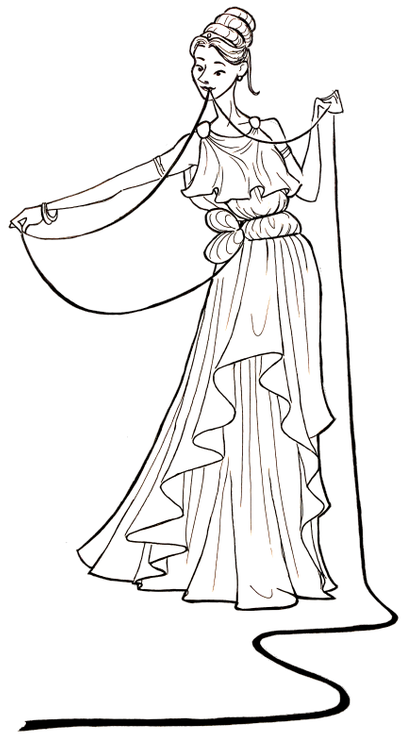

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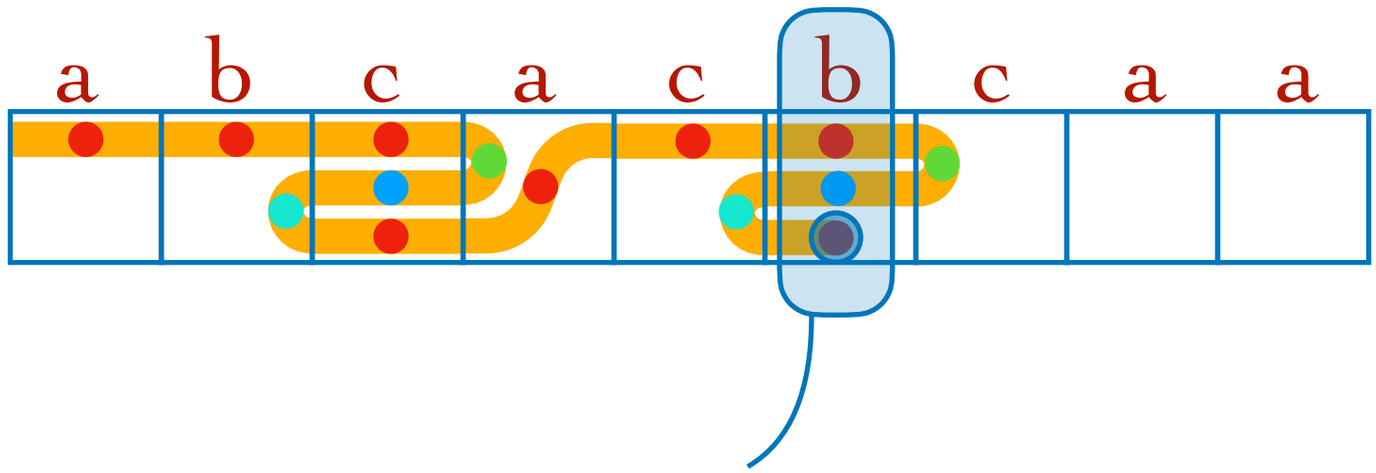

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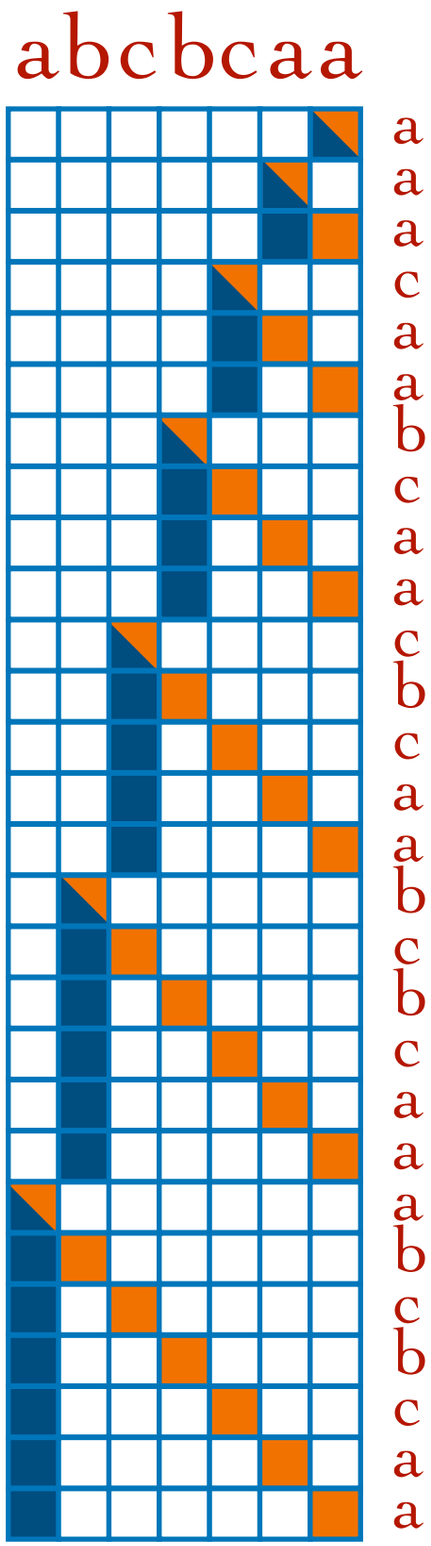
Also: A **total order** on the states guarantees that it does not loop:

- transitions are required that states increase a each update.

Example (polyregular)



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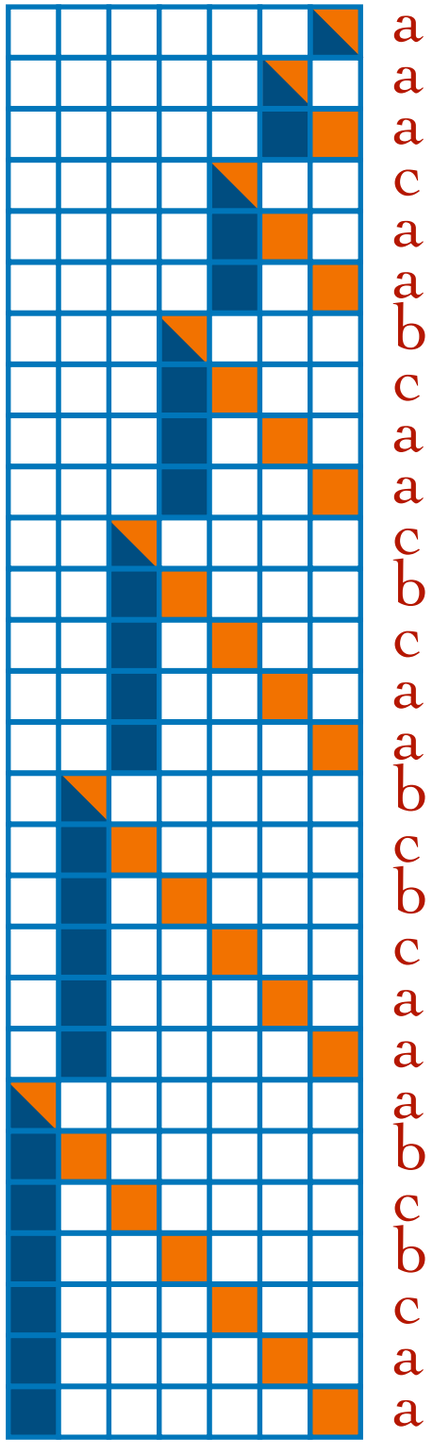

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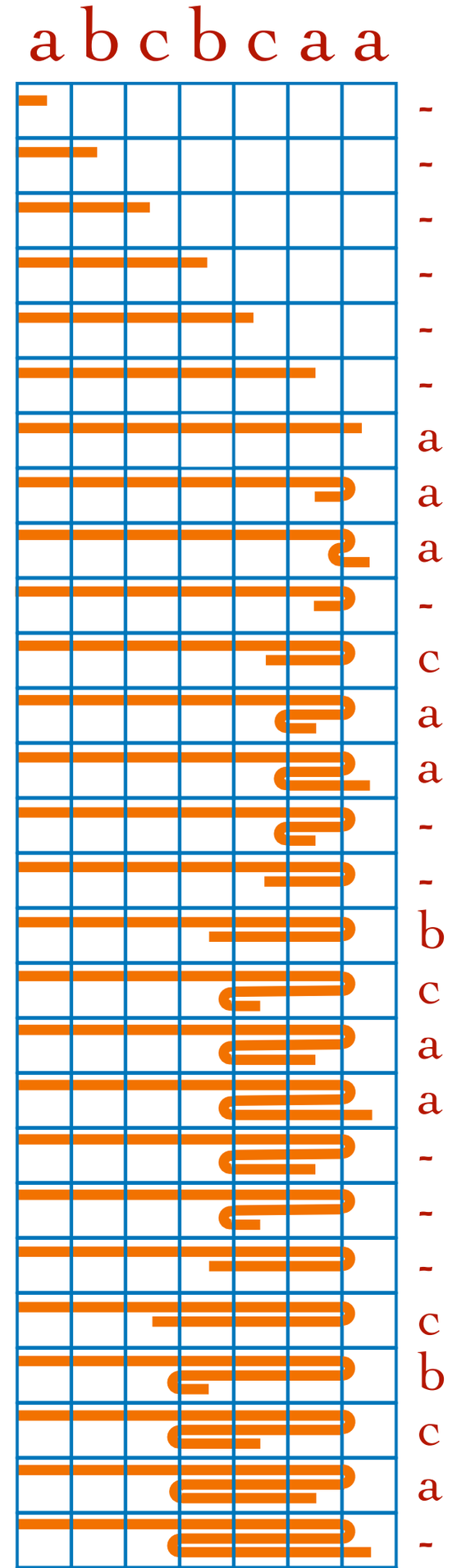

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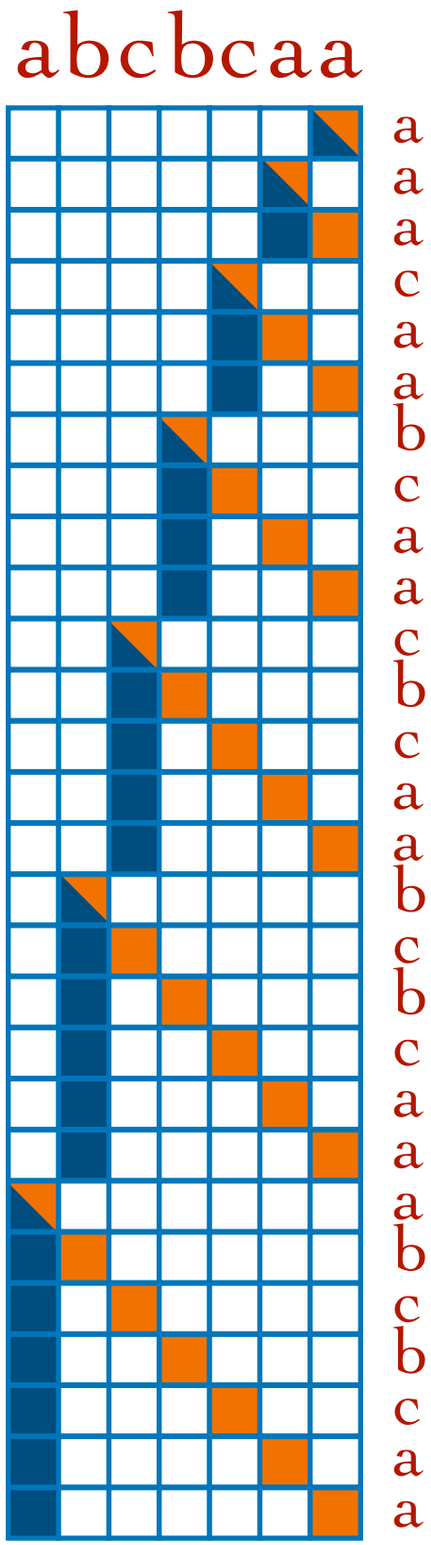

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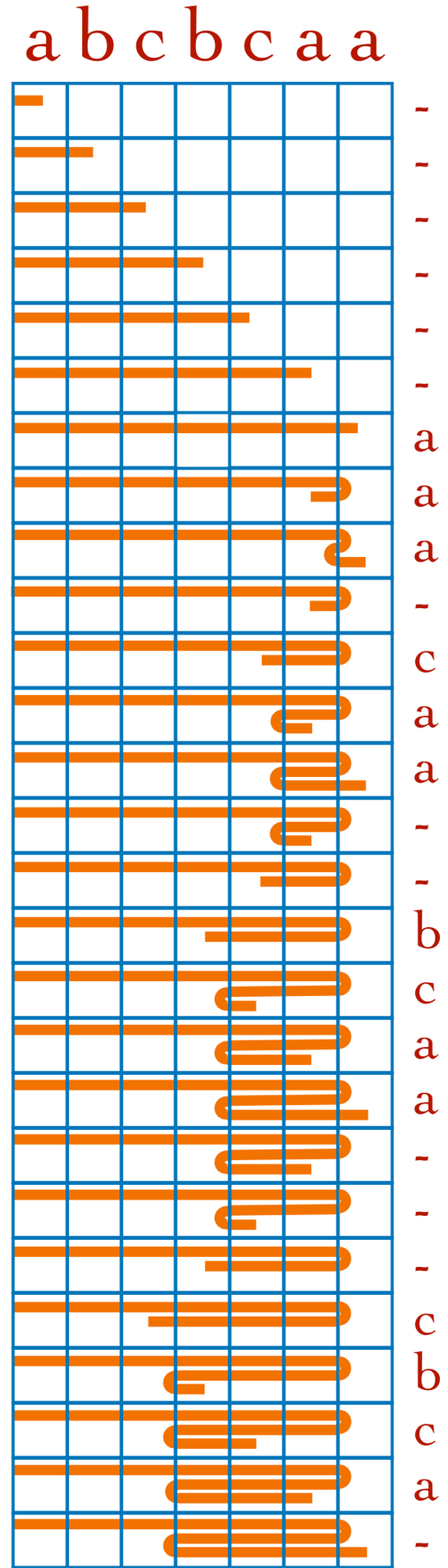
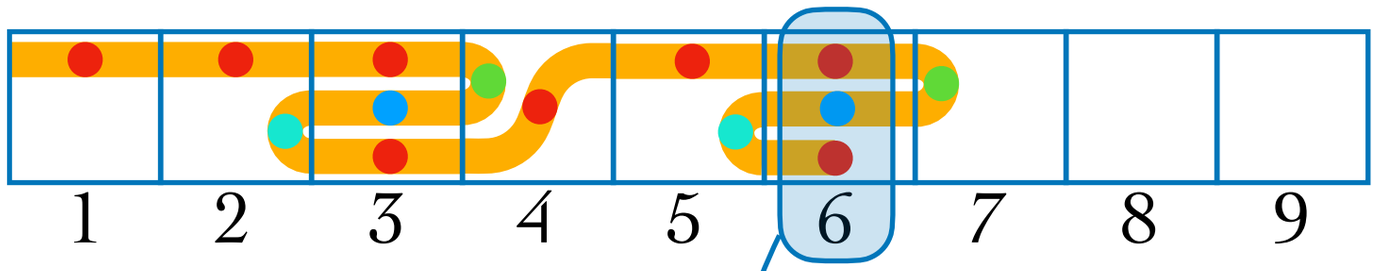


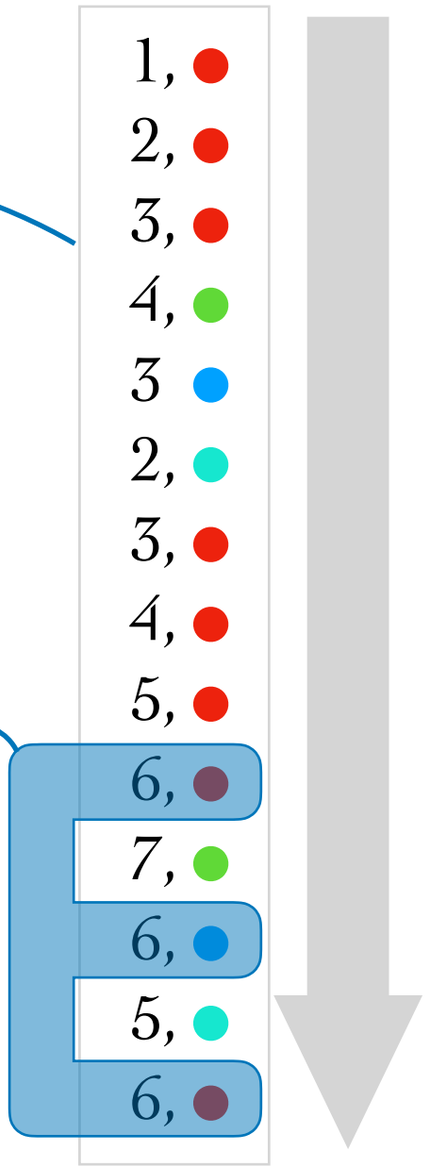
Remark: for polyregular functions, each walk need only use a bounded number of changes of direction.

From Ariadne-transducers to MSO

(Easy direction)

A configuration is a stack of pairs (called global stack):
(position,state)





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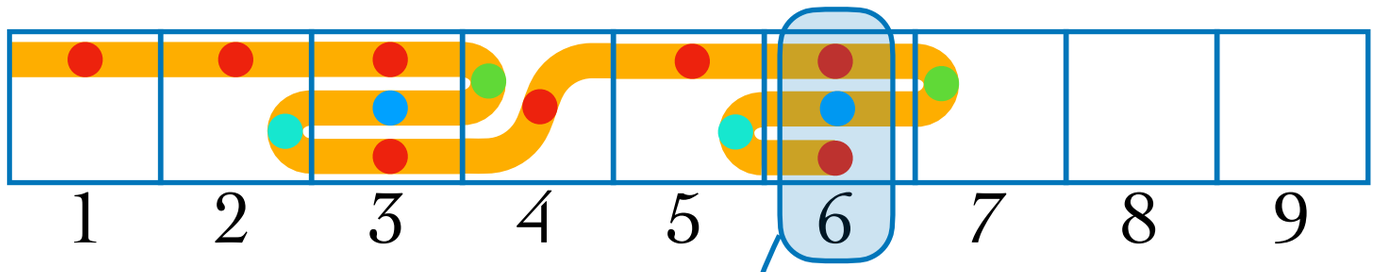

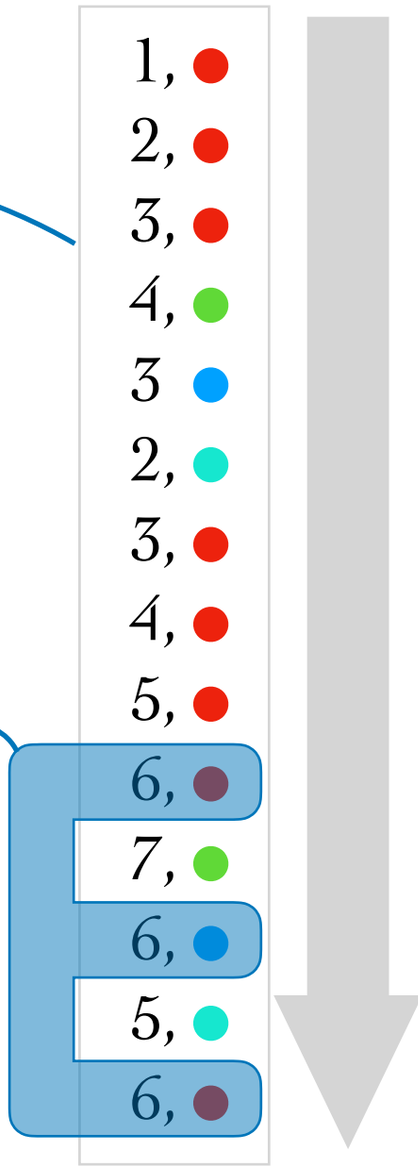

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Remark: Global stacks ordered by prefix produce a forest of linear height and bounded rank.

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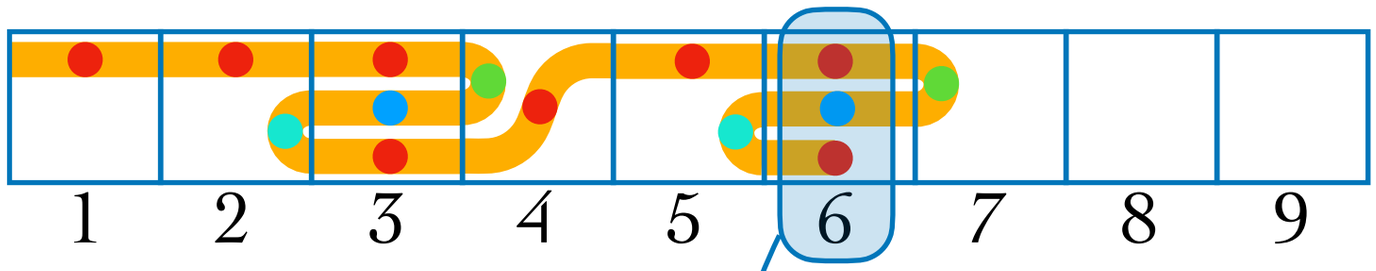

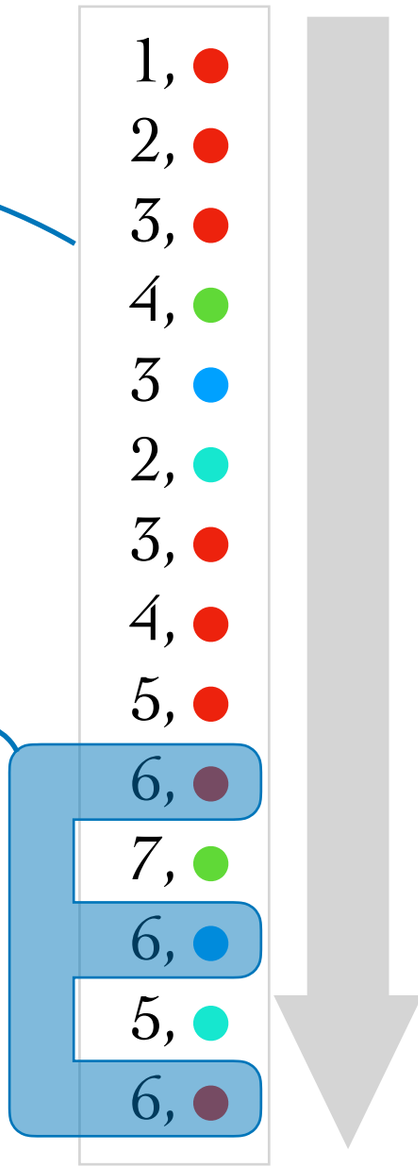

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Remark: Configurations can be represented by tuples of sets (=colorings).

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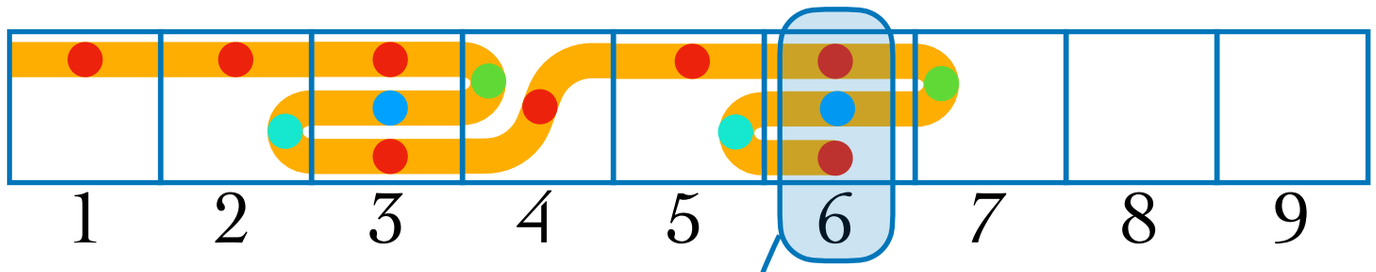
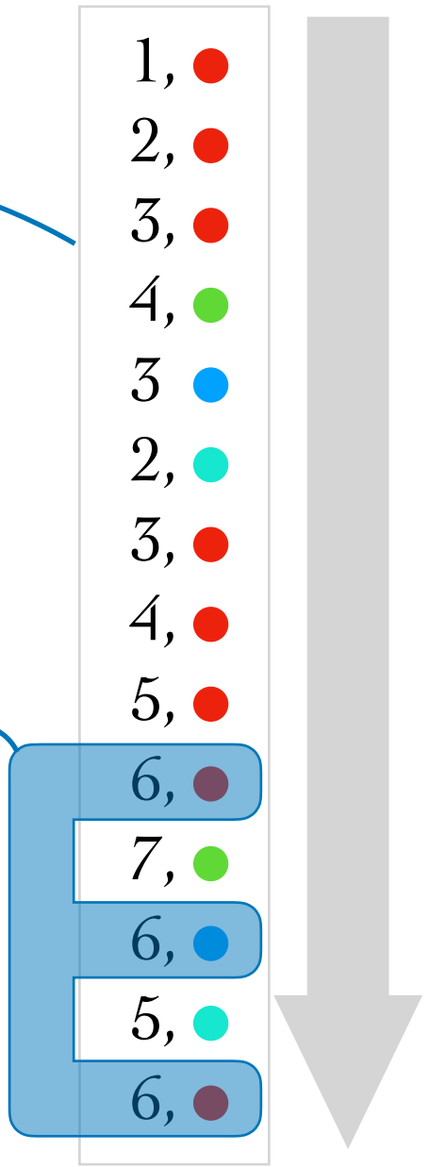




The local stack is its restriction to the head position.

Remark: Global stacks ordered by prefix produce a forest of linear height and bounded rank.

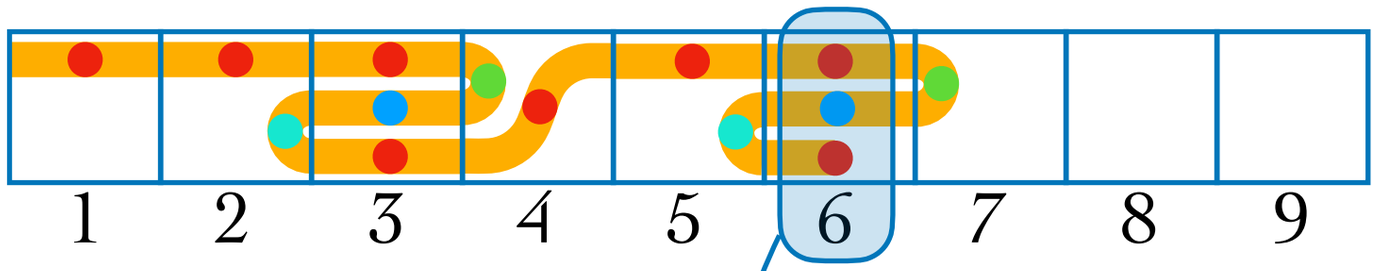
Remark: Configurations can be represented by tuples of sets (=colorings).

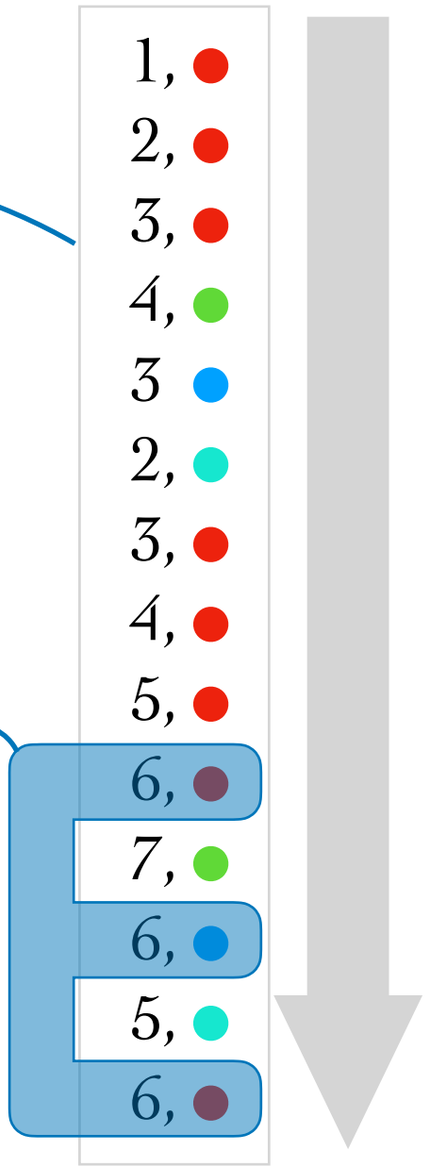
Not obvious: The set of reachable configurations is MSO-definable.

From Ariadne-transducers to MSO

(Easy direction)

A configuration is a stack of pairs (called global stack):
(position,state)





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Remark: Configurations can be represented by tuples of sets (=colorings).

Not obvious: The set of reachable configurations is MSO-definable.

Corollary: The output is MSO-set-interpretable in the input. Hence: Functions computed by Ariadne-transducers are MSO-set-interpretations.

Lambda transducers

Consider an input alphabet Σ, and an output alphabet Γ.

A **lambda-transducer** from Σ^* to Γ^*, consists of

- a simple type τ built from the base type o (using $\rightarrow$)
- a simply typed lambda-term $\mathsf{I} : \tau$,
- a simply typed lambda-term $\mathsf{F} : \tau \rightarrow \mathsf{o}$,
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$$(T_{a1} ; T_{a2} ; \ldots ; T_{an} ; F)\ I : o$$

Can be totally beta-reduced (in finitely many steps) into a term which has the form (is equivalent to):

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The **image** of u by the lambda-transducer is $b_1b_2\ldots b_{m-1}b_m$.

Lambda transducers

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Lemma: Ariadne-transducers can be effectively transformed into equivalent lambda-transducers.

The parameter k corresponds to the order of τ.

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Lemma: The inverse image of a regular language by a lambda-transducers is regular.

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Corollary: MSO-set-interpretations preserve regular language of strings by inverse image.

Conclusion

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Interpretations

Finite model theory

Simply typed
lambda-calculus

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Conjecture: Reverse holds under an exponential growth assumption.

Simply typed lambda-calculus

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