# Limits of sequences of finite structures from monadically stable classes

### Samuel Braunfeld (Charles University, Czech Academy of Sciences) Joint work with Jarik Nešetřil and Patrice Ossona de Mendez

#### May 27, 2025 Finite and Algorithmic Model Theory

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# BOREL STRUCTURES AND MODELINGS

#### Definition

A (*totally*) *Borel structure* is a structure whose domain is a standard Borel space of size  $2^{\aleph_0}$  (i.e. [0, 1] with its Borel sigma-algebra) such that every parameter-definable relation is Borel (in the product  $\sigma$ -algebra).

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#### Definition

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• We may measure higher-arity definable relations using the product measure.

- $(G_n)_{n \in \omega}$  a sequence of finite structures with  $|G_n| \to \infty$ .
- We equip each  $G_n$  with the normalized counting measure  $\mu_n$ , so  $\mu_n(\phi(\bar{x})) = |\phi(G_n)|/|G_n|^{|\bar{x}|}$ .

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 $(G_n)_{n \in \omega}$  is *FO-convergent* if  $\lim_{n \to \infty} \mu_n(\phi(\bar{x}))$  exists for every formula  $\phi(\bar{x})$ . (Sentences should be asymptotically true or false.)

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• Note an ultraproduct almost works, but is not Borel.

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• Really a purely infinitary theorem, applied to the ultraproduct.

# Sketching the proof



# Sketching the proof



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# SKETCHING THE PROOF



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Let G be an edge-Lebesgue graph (random-free graphon). Then there is a edge-Borel graph G' equivalent to G removing all induced subgraphs that appear with measure 0.

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- Similar to the previous construction, but
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    we don't worry about making *B* ≻ *N*

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- Can we obtain the mass transport principle (double-counting edges between measurable sets)? This needs a different topology on our modeling.
- Modeling limits for sequences from monadically NIP/dependent classes? (Ongoing work with Pierre Simon)

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