

# Separating Regular Relations

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# REGULAR RELATIONS

# Regular Relations

**Intuition:** Regular languages over  $\Sigma^* \times \dots \times \Sigma^*$  ( $n$  times)

**Accepted by:** Synchronous automata over  $(\Sigma \cup \{ \perp \})^n$

# Synchronous Automata

$$\bar{w}_1 = a \ b \ a \ b$$

$$\bar{w}_2 = a \ a \ a \ b \ b \ b$$

# Synchronous Automata

$$\begin{aligned}\bar{w}_1 &= a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 &= a \ a \ a \ b \ b \ b\end{aligned}$$

# Synchronous Automata


$$\begin{aligned}\bar{w}_1 &= a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 &= a \ a \ a \ b \ b \ b\end{aligned}$$

$\uparrow$   
 $q_0$

Initial state

# Synchronous Automata

$$\begin{aligned}\bar{w}_1 &= a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 &= a \ a \ a \ b \ b \ b\end{aligned}$$

$$q_1 = \delta(q_0, (a, a))$$


# Synchronous Automata

$$\begin{array}{l} \bar{w}_1 = a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 = a \ a \ a \ b \ b \ b \end{array}$$



$$q_2 = \delta(q_1, (b, a))$$



# Synchronous Automata

$$\begin{array}{l} \bar{w}_1 = a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 = a \ a \ a \ b \ b \ b \end{array}$$

$$\begin{array}{c} \uparrow \\ q_3 = \delta(q_2, (a, a)) \end{array}$$

# Synchronous Automata

$$\begin{array}{l} \bar{w}_1 = a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 = a \ a \ a \ b \ b \ b \end{array}$$

$$\begin{array}{c} \uparrow \\ q_4 = \delta(q_3, (b, b)) \end{array}$$


# Synchronous Automata

$$\begin{array}{l} \bar{w}_1 = a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 = a \ a \ a \ b \ b \ b \end{array}$$

$$\begin{array}{c} \uparrow \\ q_5 = \delta(q_4, (\perp, b)) \end{array}$$

# Synchronous Automata

$$\begin{aligned}\bar{w}_1 &= a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 &= a \ a \ a \ b \ b \ b\end{aligned}$$

$$q_6 = \delta(q_5, (\perp, b))$$


# Synchronous Automata

$$\begin{array}{l} \bar{w}_1 = a \ b \ a \ b \ \perp \ \perp \\ \bar{w}_2 = a \ a \ a \ b \ b \ b \end{array}$$

$$q_6 = \delta(q_5, (\perp, b))$$

The pair  $(\bar{w}_1, \bar{w}_2)$  is **accepted** by the synchronous automata  
iff  $q_6$  is a final state

# Regular Relations

A relation  $L \subseteq \Sigma^* \times \Sigma^*$  is **regular** iff there exists a synchronous automata that accepts precisely the pairs of words in  $L$

# Examples of Regular Relations

**Equal-length relation:** Pairs  $(\bar{w}_1, \bar{w}_2)$  of words of the same length

**Prefix relation:** Pairs  $(\bar{w}_1, \bar{w}_2)$  of words such that  $\bar{w}_1$  is a prefix of  $\bar{w}_2$

**Edit distance:** Pairs  $(\bar{w}_1, \bar{w}_2)$  of words such that  $\bar{w}_1$  is at edit distance at most  $k$  from  $\bar{w}_2$ , for a fixed  $k > 0$

# Examples of Non-Regular Relations

**Suffix relation:** Pairs  $(\bar{w}_1, \bar{w}_2)$  of words such that  $\bar{w}_1$  is a suffix of  $\bar{w}_2$

**Subsequence relation:** Pairs  $(\bar{w}_1, \bar{w}_2)$  of words such that  $\bar{w}_1$  is a  
(scattered) subsequence of  $\bar{w}_2$



**SEPARABILITY /  
DEFINABILITY**

# Separability

Given two regular relations  $L_1$  and  $L_2$ , is there a “simple” regular relation  $L$  such that:

$$L_1 \subseteq L \quad \text{and} \quad L_2 \cap L = \emptyset?$$

# Definability

Given a regular relation  $L$ , is there a “simple” regular relation  $R$  such that  $L = R$ ?

# A Simple Class of Regular Relations

**Recognizable relations:** Finite unions of products of regular languages

The pair  $(\bar{w}_1, \bar{w}_2)$  is **accepted** by the recognizable relation

$$\bigcup_{i \leq n} L_i^1 \times L_i^2$$

iff  $\bar{w}_1$  is accepted by  $L_i^1$  and  $\bar{w}_2$  is accepted by  $L_i^2$ , for some  $i \leq n$

# The Definability Problem

Theorem (B., Hong, Le, Lin, Niskanen, 2019)

Given a regular relation  $L$ , is there a recognizable relation  $R$  such that  $L = R$ ?

This problem is PSPACE-complete  
(NLOGSPACE-complete if  $L$  is deterministic)

# The Separability Problem

Given two regular relations  $L_1$  and  $L_2$ , is there a recognizable relation  $L$  such that:

$$L_1 \subseteq L \quad \text{and} \quad L_2 \cap L = \emptyset?$$

The decidability of this problem is **OPEN!**

# REGULAR GRAPHS

# Regular Graphs

Those of the form:

$$\mathcal{G}_L = (\Sigma^*, L),$$

where  $L$  is a regular relation over  $\Sigma^* \times \Sigma^*$



# Separability and Colorability

Theorem (B., Figueira, Morvan, 2023)

There are polynomial-time reductions between the following problems:

- (1) Given two regular relations  $L_1$  and  $L_2$ , is there a recognizable relation  $L$  such that:

$$L_1 \subseteq L \quad \text{and} \quad L_2 \cap L = \emptyset?$$

- (2) Given a regular relation  $L$ , is the regular graph  $\mathcal{G}_L$  finitely colorable with every color defining a regular language?

# Finite Colorability

For regular graphs

Finite colorability with regular colors: OPEN

Finite colorability: OPEN

# On Cliques

For regular graphs

Existence of infinite cliques: Decidable  
(Kuske, Lohrey, 2010)

Existence of arbitrarily large cliques: OPEN

# $k$ -Colorability

Theorem (B., Figueira, Morvan, 2023)

The following problem is undecidable for each fixed  $k > 1$ :

Given a regular relation  $L$ , is the regular graph  $\mathcal{G}_L$   $k$ -colorable with every color defining a regular language?

This problem is also undecidable  
without the restriction of colors being regular sets  
(Köcher, 2014)

# Separability and $k$ -Colorability

Proposition (B., Figueira, Morvan, 2023)

There is a polynomial-time reduction from the first to the second problem:

- (1) Given a regular relation  $L$ , is the regular graph  $\mathcal{G}_L$   $k$ -colorable with every color defining a regular language?
- (2) Given two regular relations  $L_1$  and  $L_2$ , is there a recognizable relation  $L = \bigcup_{i \leq k} L_i^1 \times L_i^2$  such that:

$$L_1 \subseteq L \quad \text{and} \quad L_2 \cap L = \emptyset?$$

# Conclusion

Corollary (B., Figueira, Morvan, 2023)

The following problem is undecidable for every fixed  $k > 1$ :

Given two regular relations  $L_1$  and  $L_2$ , is there a recognizable relation  $L = \bigcup_{i \leq k} L_i^1 \times L_i^2$  such that:

$$L_1 \subseteq L \quad \text{and} \quad L_2 \cap L = \emptyset?$$