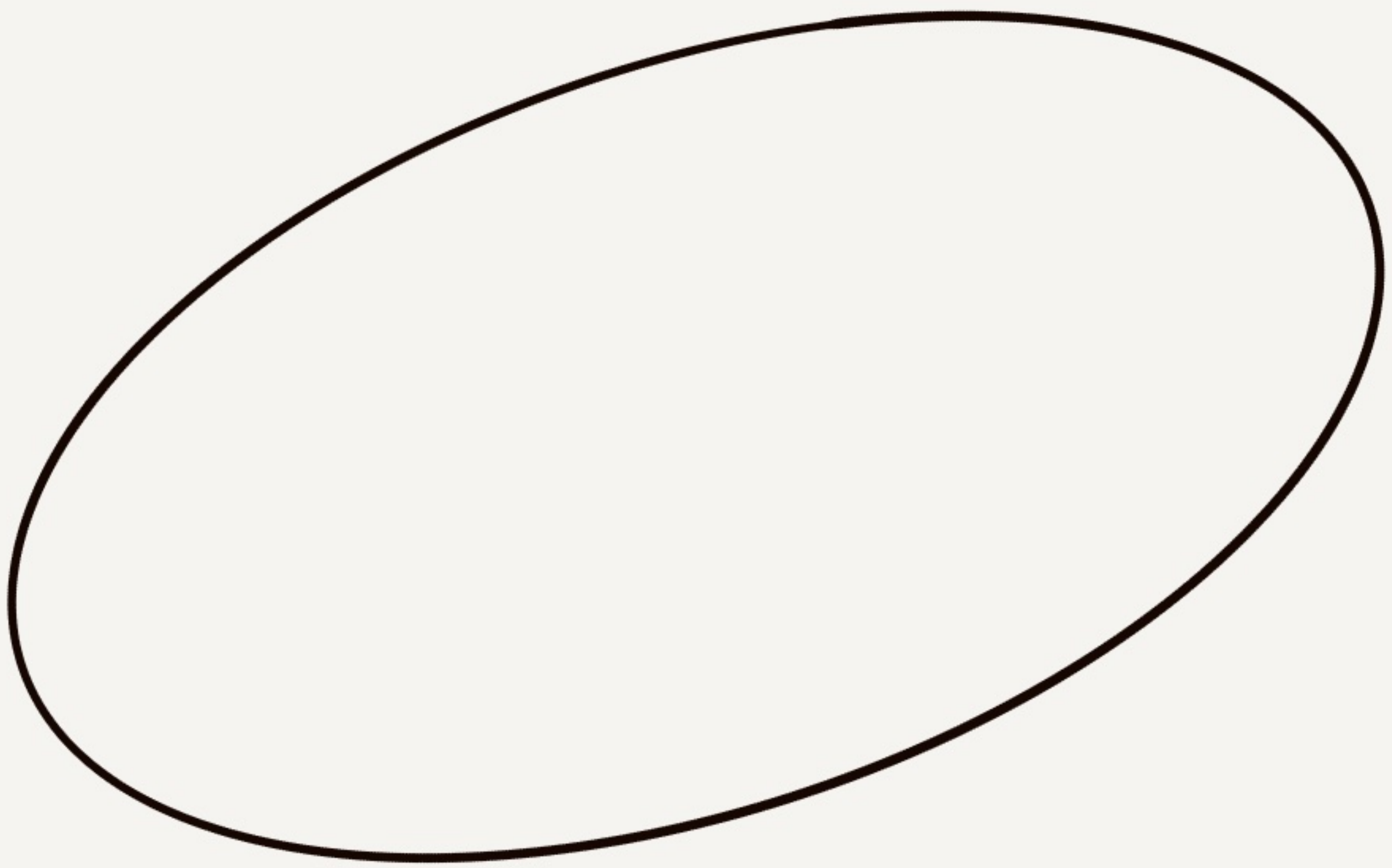
ON CERTIFYING THE CHROMATIC NUMBER OF SPARSE RANDOM GRAPHS

Albert Atserias
UPC Barcelona

$G = G(n, p = \frac{d}{n})$



n : number of vertices

m : number of edges

d : expected avg. deg.

q : number of colors

Problem statement:

Certify / Prove that $\chi(G) > q$ a.a.s.

$G = G(n, \frac{d}{n})$

can be improved to $2q\ln(q) - \ln(q)$

First moment:

If $d \geq 4q\ln(q)$, then $\Pr_G[\chi(G) > q] \xrightarrow[n\to\infty]{} 1$

Chernoff + union bound:

far from q-colorable

If $d \geq \frac{4q\ln(q)}{\varepsilon^2}$, then

$$\Pr_G\left[\begin{array}{l}\text{any } q\text{-coloring of } G \\ \text{leaves } \geq (1-\varepsilon)\frac{m}{q} \\ \text{monochromatic edges}\end{array}\right] \xrightarrow[n\to\infty]{} 1$$

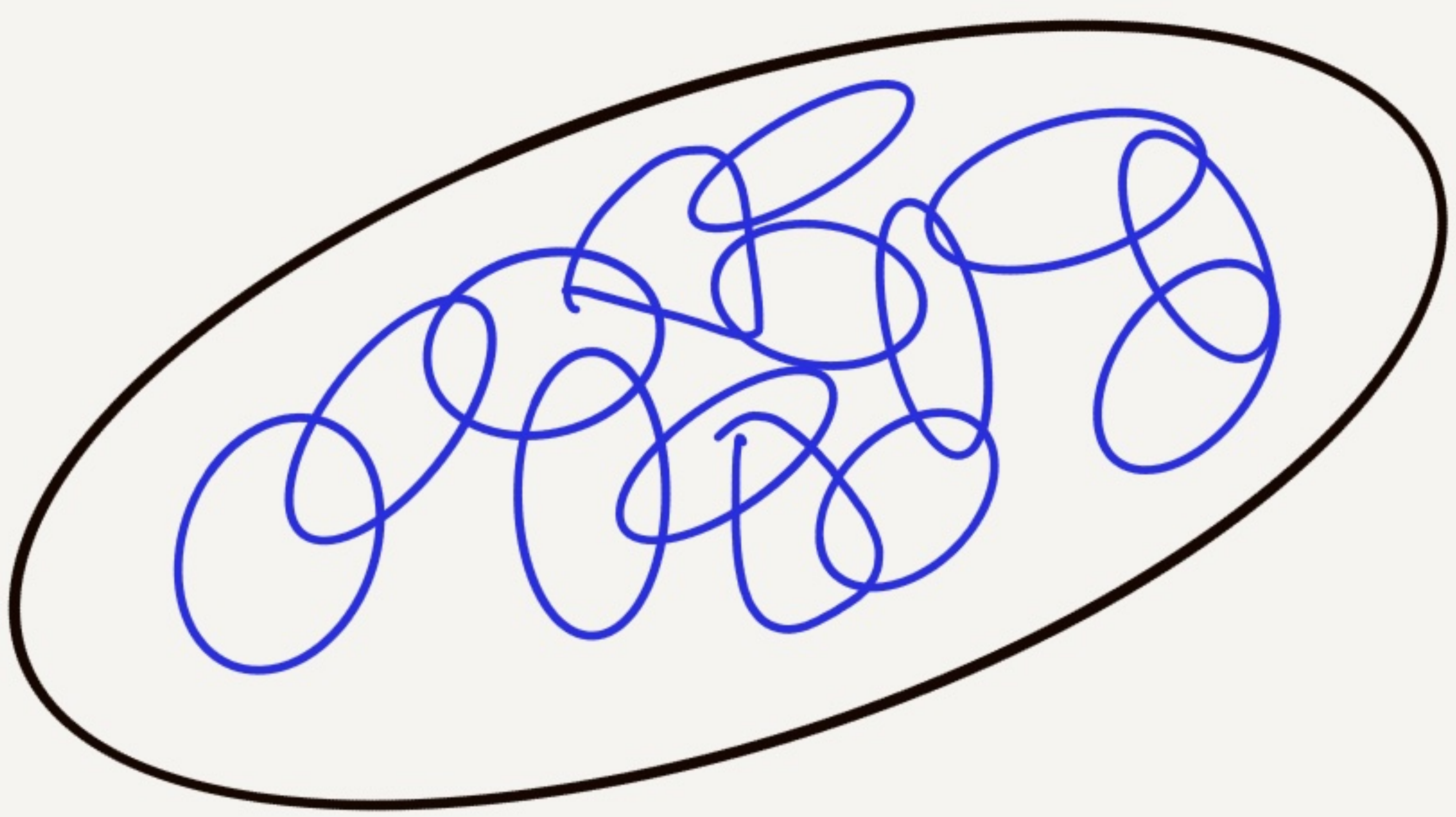
Logic Certificates of $\chi(G) > q$

Naive idea #1: Fix $k \geq q$.

check if $\chi(G[A]) > q$ for some $A \subseteq [n]$, $|A| \leq k$

Less naive idea #2: Fix $k \geq q$.

sliding window of size $\leq k$



the k-consistency algorithm for q-coloring.

(next slide)

k-consistency algorithm

1) for each $A \in \binom{[n]}{\leq k}$ collect
$$c(A) = \{ h : [n] \to [q] \mid h \text{ properly colors } G[A] \}$$

2) if there exist $A \in \binom{[n]}{<k}$, $h \in C(A)$, $v \in [n] \setminus A$ s.t.
$$h \cup \{v \mapsto 1\}, \dots, h \cup \{v \mapsto q\} \notin C(A \cup \{v\})$$
then remove h from $C(A)$

3) repeat until all $C(A)$ stabilize (they must!)

4) if ~~all~~ some $C(A)$ become empty
then $\boxed{\chi(G) > q \quad \text{\#CERTIFIED\#}}$

How good is K-consistency?

- enough to certify $\chi(G) > 2$ even with $K = 3$
- enough to certify $\chi(G) > q$ on treewidth $K > q$.
- not immediately fooled by local unicyclicity.

Question But is it any good for $G = G(n, \frac{d}{n})$?

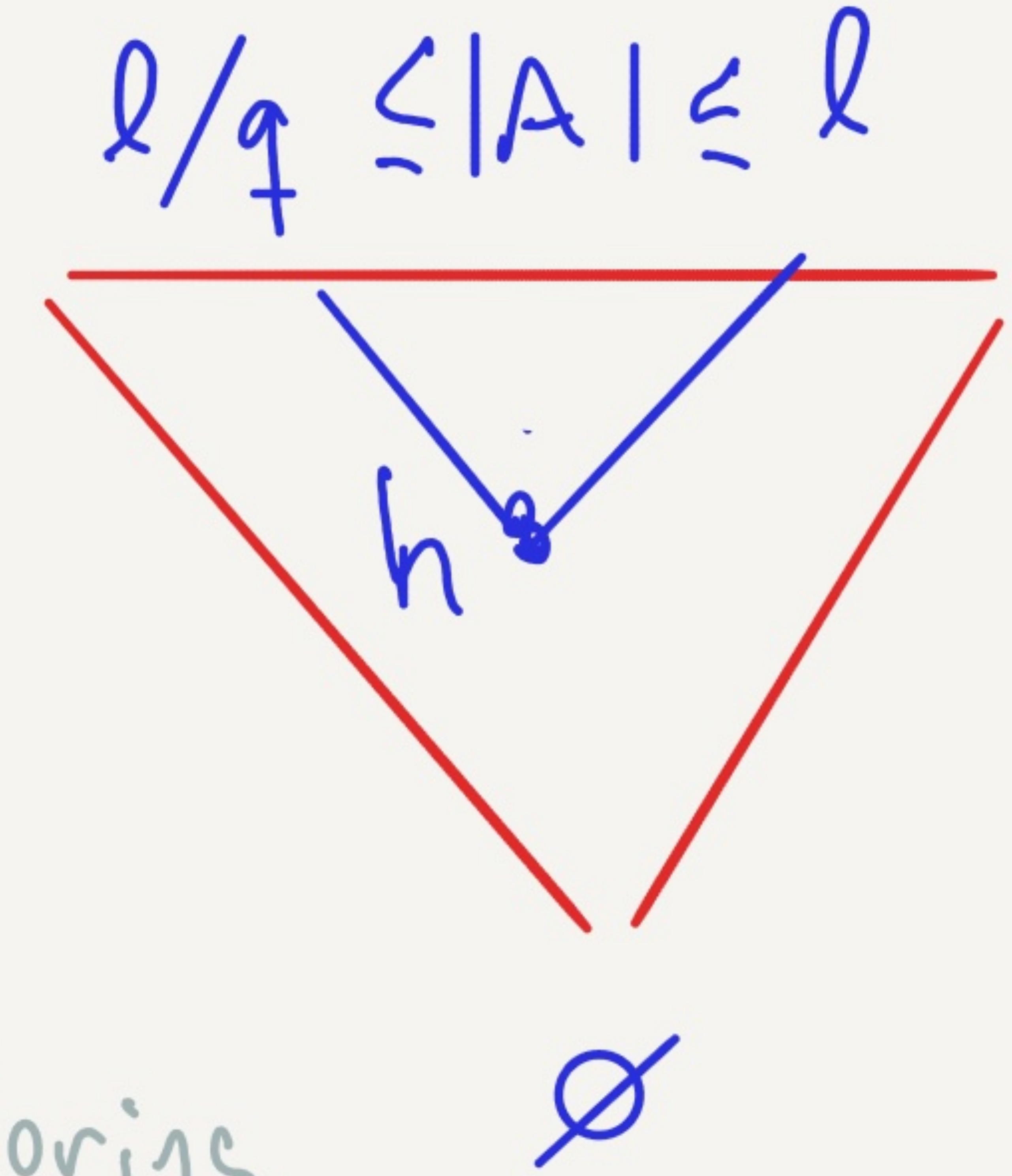
spoiler: understood and answer is NO. (requires $K = \Omega_d(n)$)

Refutations

(1) $$\frac{}{h} \quad \left[\begin{array}{l} h \text{ not a proper coloring} \\ |\mathrm{Dom}(h)| \le K \end{array}\right]$$

(2) $$\frac{h \cup \{v \mapsto 1\} \quad \cdots \quad h \cup \{v \mapsto q\}}{h} \quad \left[\, |\mathrm{Dom}(h)| < K \,\right]$$

(3) $$\frac{h}{h'} \quad \left[\, h \subseteq h' \,\right]$$

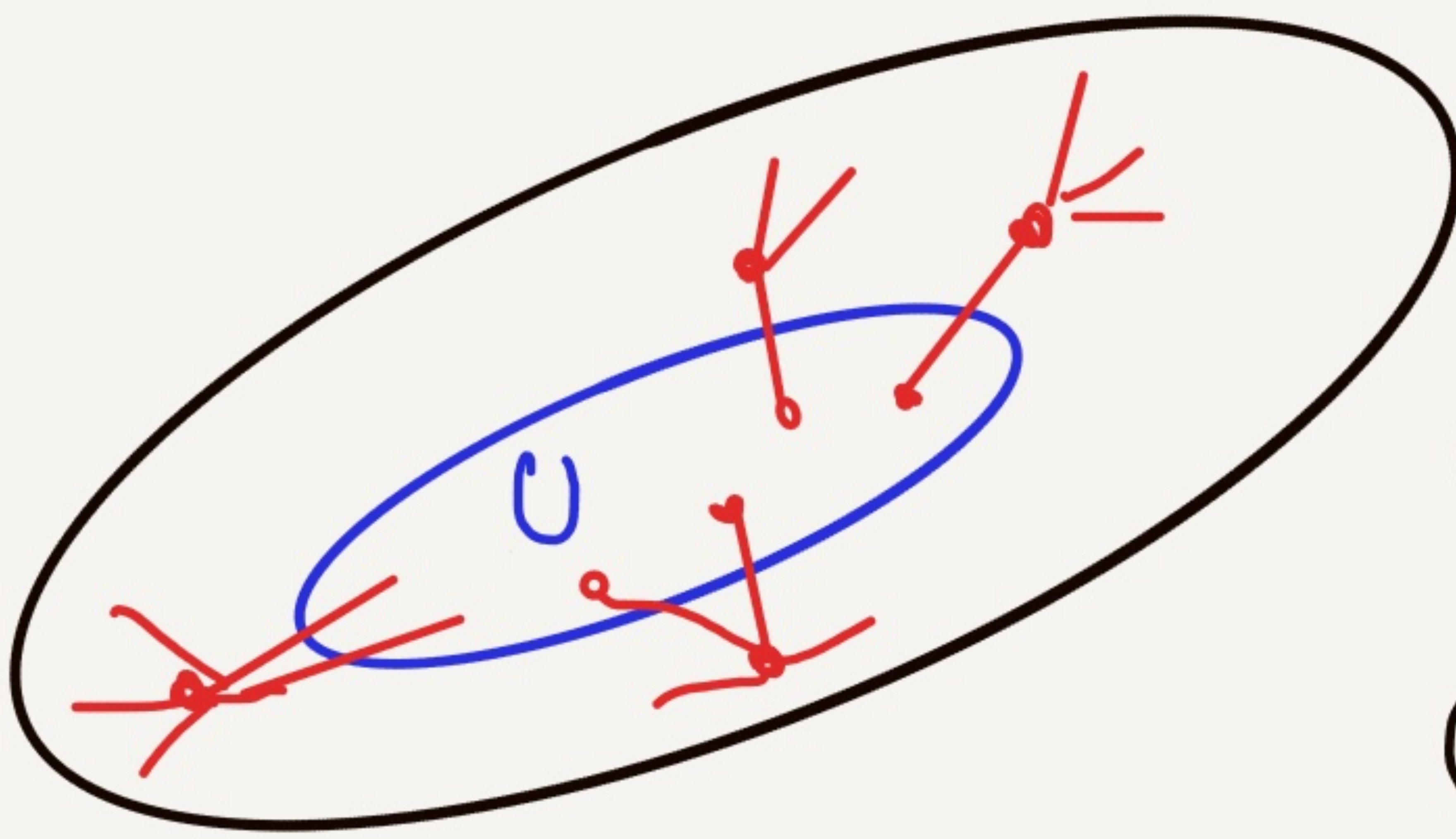


Goal: Derive $\emptyset$.

"empty coloring cannot be extended to a proper coloring"

Sparsity: (ℓ, ε)-sparse [CS'88]

Fact: For $G \sim G(n, \frac{d}{n})$ a.a.s. every $S \subseteq [n]$ with $|S| \le \ell$ has $|E(S)| \le (1+\varepsilon)|S|$ where $\ell \ge \Omega_d(n)$ and $\varepsilon = o(1)$.



(1) sparsity & $|U| \le \ell \Rightarrow \chi(G[U]) \le 3$

(2) sparsity & $|U| \le \ell \Rightarrow |\partial U| \ge c_d |U|$

$$\partial U := \{ v \notin U : |N(v) \cap U| < q \}$$

Algebraic reformulation & generalization

Variables:

$X(h)$: where $h : A \to [q]$ with $A \subseteq [n]$, $|A| \leq k$.

Equations:

$$E_k^q(G): \begin{cases} X(h) = 0 & \text{if } h : A \to [q] \text{ is not a proper } q\text{-coloring of } G[A]. \\ \sum_{i=1}^{q} X(h \cup \{u \mapsto i\}) = X(h) & \text{for all } h : A \to [q],\ A \subseteq [n],\ |A| < k,\ u \in [n] \setminus A \\ X(\emptyset) = 1 \end{cases}$$

· for now, variables range over the Boolean monoid $\mathbb{B} = (\{0,1\}, \vee, 0)$

Algebraic Certificates of $\chi(G) > q$

- $E^q_K(G)$ over fields ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_q$, …) → linear algebra
- $E^q_K(G)$ over $\mathbb{Q}^{\geq 0}$. → linear programming

---

$[Ax = b]$ infeasible

iff

$[A^T y = 0, \; b^T y = 1]$ feasible

certificates/proofs of $\chi(G) > q$

Nullstellensatz

$[Ax = b, \; x \geq 0]$ infeasible

iff

$\begin{bmatrix} A^T y + I z = 0 \\ b^T y = -1 \\ z \geq 0 \end{bmatrix}$ feasible

Sherali-Adams

linear algebra (NS)

linear programing (LP)

Question: But are LA and LP any good for $G(n, \frac{d}{n})$?

For SA: surprinsingly open even for $k = O(1)$.

For NS: solved only recently:

Thm: [CdRNPR'23] For $G \sim G(n, \frac{d}{n})$ a.a.s. the system $E_k^q(G)$ is feasible over any field even for $k = \Omega_d(n)$.

Semialgebraic formulation [open : major]

$$\mathrm{SE}_k^q(G)\quad\left[\begin{array}{l} X(h) = 0 \\ \sum_{i=1}^{q} X(h \cup \{u \mapsto i\}) = X(h) \\ X(\emptyset) = 1 \\ M \succcurlyeq 0 \end{array}\right.$$

M is positive semidefinite

$$M(g,h) = \begin{cases} X(g \cup h) & \text{if } g \text{ and } h \text{ are compatible} \\ 0 & \text{otherwise.} \end{cases}$$

maps $g : A \to [q]$, $h : B \to [q]$ with $|A \cup B| \leq k$

Soundness of $SE_k^q(G)$

Assume $h^*: [n] \to [q]$ is a proper q-coloring of G.

Then

$$x(h) := \begin{cases} 1 & \text{if } h \subseteq h^* \\ 0 & \text{if } h \not\subseteq h^* \end{cases}$$

is feasible.

$$M(g,h) = U(g) \cdot U(h) = U^T U \succcurlyeq 0$$

for

$$U(g) = \begin{cases} 1 & \text{if } g \subseteq h^* \\ 0 & \text{if } g \not\subseteq h^* \end{cases}$$

in this case, even rank 1 PSD.

A positive result: SDP certificates

Theorem: If $SE_2^q(G)$ is feasible, then there exists $h:[n]\to[q]$ that leaves at most

$$\left(1-\frac{c_0}{q^3}\right)\frac{m}{q}$$

monochromatic edges in G.

based on [H'08]

recall $\frac{4q\ln(q)}{\varepsilon^2}$ for $(1-\varepsilon)\frac{m}{q}$

Corollary: If $d \geq \frac{4q^7\ln(q)}{c_0^2}$ and $G=G(n,\frac{d}{n})$ then the SDP $SE_2^q(G)$ is infeasible a.a.s.

feasibility of dual #CERTIFIED#

Smaller window : enter $\vartheta(\bar{G})$

Thm [BKM'19] improves [C-O'08]

If $d > 4q^2$ and $G \sim G_{reg}(n,d)$
then the SDP $SE_2^q(G)$ is infeasible a.a.s.

Note 1: Analysis via Lovasz theta.

Note 2: Remains open for:

$$d \in [2q\ln(q), 4q^2]$$

even for

$$k = 2 \,!\ \text{ in } SE_k^q(G)$$

the Excuse

Thm [AD'22]

If a PCSP$(\mathbb{A}, \mathbb{B})$ is solvable in $o(n)$-width then it has WNU's of all arities $3, 4, 5, \ldots$

WNU: $f : \mathbb{A}^r \to \mathbb{B}$ homomorphism such that

$$f(x, y, \ldots, y) \simeq f(y, x, y, \ldots, y) \simeq \cdots \simeq f(y, \ldots, y, x)$$