Some recent advances in logic and graph property testing

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Joint work with Frederik Harwath, Noleen Köhler, Pan Peng, Joseph Stimpson

FMT Workshop
Les Houches
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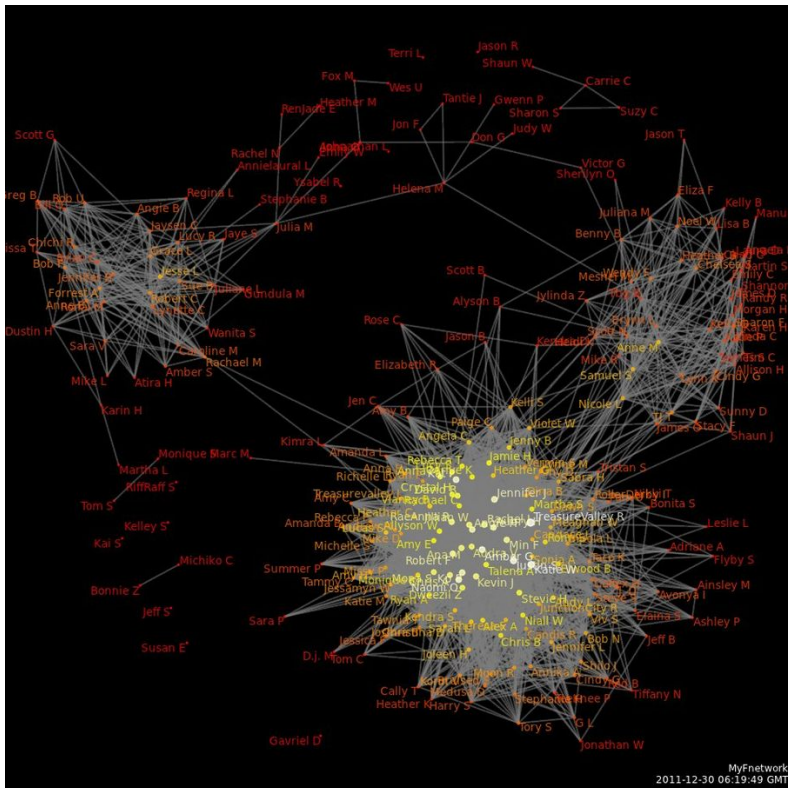
Welcome to the quest...

Contents

Property testing: motivation

‘Efficiency’ when the data set is huge:

Even reading the whole input just once can be too expensive.



Data visualisation of Facebook relationships

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Aim: Sublinear algorithms with local access to the input.

Graphs & logics

- We consider finite, simple, undirected graphs $G = (V(G), E(G))$.
- Class $\mathcal{C}$ of graphs has bounded degree, if there is a constant $d \in \mathbb{N}$ such that all graphs in $\mathcal{C}$ have degree $\leq d$.
- We use n to denote the number of vertices of G.
- All graph classes are closed under isomorphism, and a graph class is sometimes called a property.

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MSO := Monadic second-order logic =
first-order logic + quantification over subsets of the universe

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Algorithms with oracle access

From now on: All graphs have degree $\leq d$.

- Input: the number n of vertices of G, and
- Oracle access to G
  - *Query:* v, for $v \in V(G)$
  - *Answer:* the 1-neighbourhood of vertex v
- The running time $=$ running time w.r.t. n.
- The query complexity $=$ number of oracle queries w.r.t. n.

We use the RAM model of computation.

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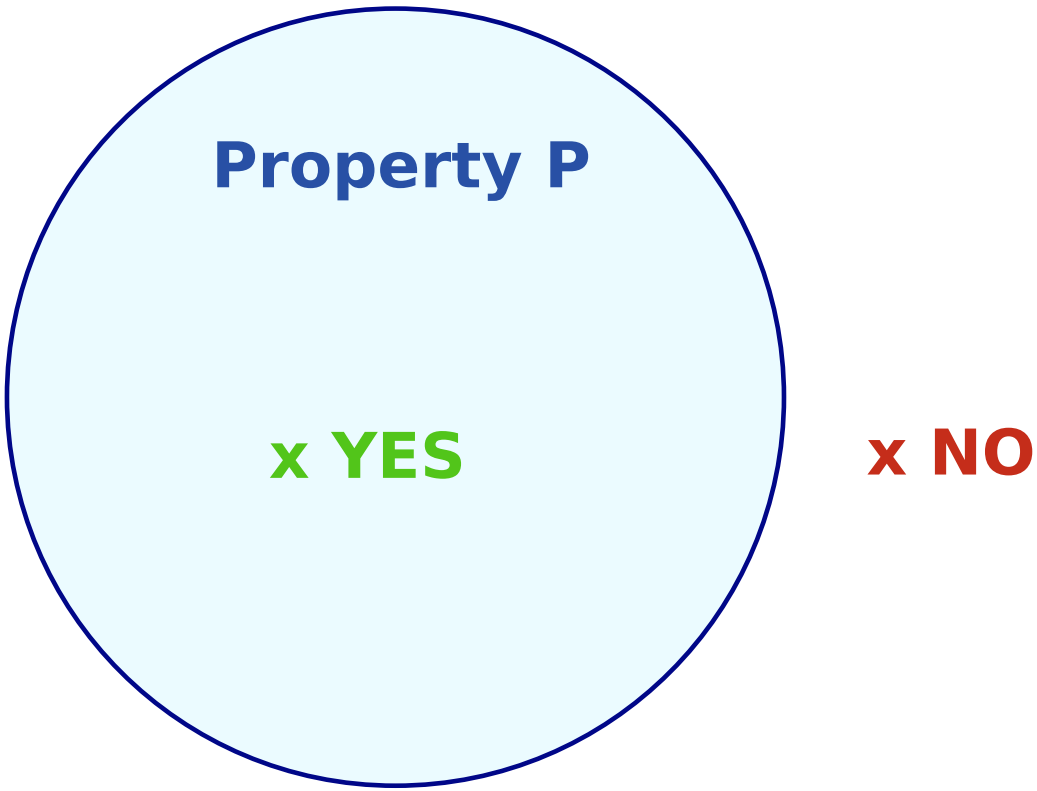
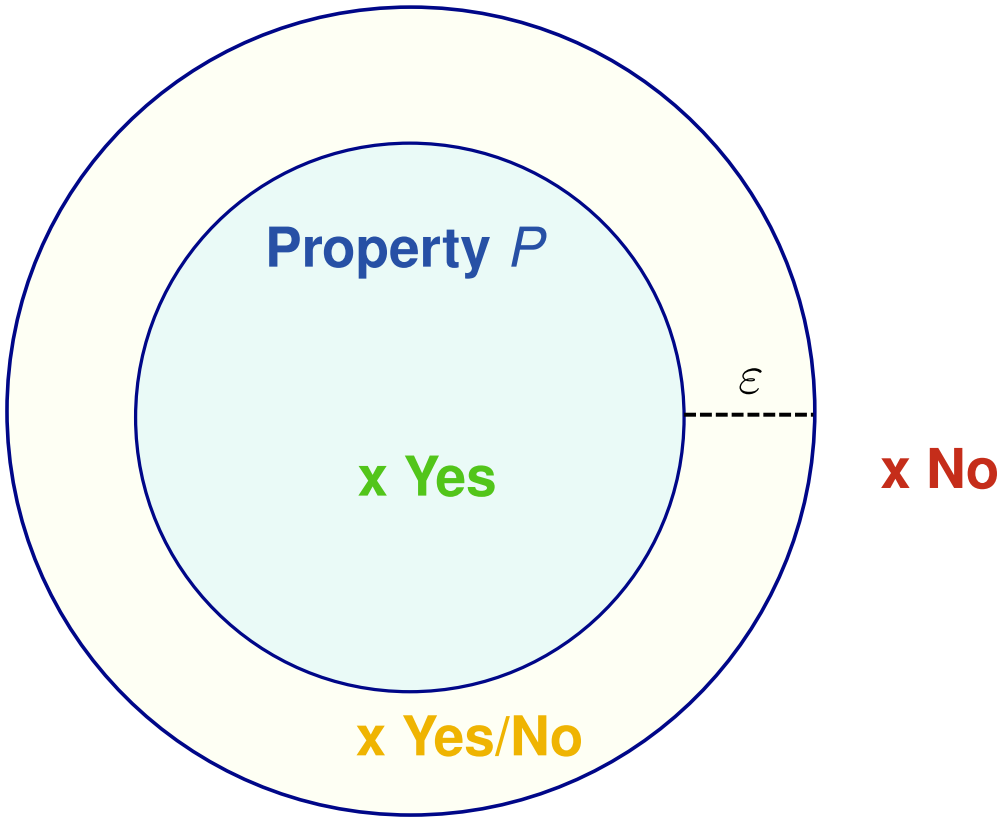


Property Testing = relaxation of decision problems



Distance (d-bounded degree model)

- Let $\varepsilon \in [0,1]$.
  Graphs G and H, both on n vertices, are ε-close, if we can make them isomorphic by modifying up to εdn edges of G or H.

  Edge modification = insertion/deletion

- If G, H are not ε-*close*, then they are ε-far.

- A graph G is ε-close to a class $\mathcal{C}$ if G is ε-close to some $H \in \mathcal{C}$.
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Testability

Let $\mathcal{P}$ be a property.
An ε-tester for $\mathcal{P}$ is a probabilistic algorithm that, given oracle access to G and given $n := |V(G)|$ as input, does the following:

1. If $G \in \mathcal{P}$, then the tester accepts with probability $\geq \frac{2}{3}$,
2. if G is ε-far from $\mathcal{P}$, then the tester rejects with probability $\geq \frac{2}{3}$.

$\mathcal{P}$ is uniformly testable, if for each ε there is an ε-tester for $\mathcal{P}$ with constant query complexity.

$\mathcal{P}$ is (non-uniformly) testable, if for each ε and each n, there is a tester for $\mathcal{P}_n := \{G \in \mathcal{P} : |V(G)| = n\}$ with constant query complexity (independent of n).

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Examples

On bounded degree graphs:

Uniformly testable with constant query complexity and running time:

- k-(edge-)connectivity
- being Eulerian
- subgraph-freeness
- induced subgraph-freeness

Not testable with constant query complexity:

- Bipartiteness, colourability
- Expander graphs
- Hamiltonicity

[Goldreich and Ron 2002; Yoshida and Ito 2010]

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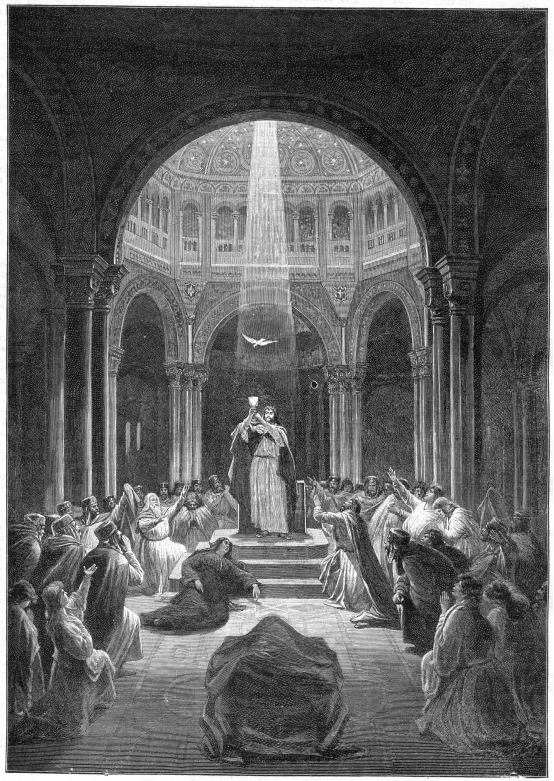
Bounded-degree model: Holy Grail I



Open problem (Holy Grail I)

Characterise the properties that are (non-uniformly) testable with constant query complexity.

Holy Grail I: what is known

Theorem (Newman and Sohler, 2013 (Benjamini, Shramm, Shapira, Elek))

Every hyperfinite property is non-uniformly testable.

Theorem (Ito, Khoury, Newman 2020)

Characterisation of 1-sided error non-uniformly testable monotone/hereditary graph properties.

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Non-testable property (proof idea)

Find an FO-formula φ whose models are expander graphs.

$(G_i)_{i\in\mathbb{N}}$: zig-zag construction [Rheingold, Vadhan, Wigdersen, Ann. of Math., 2002]

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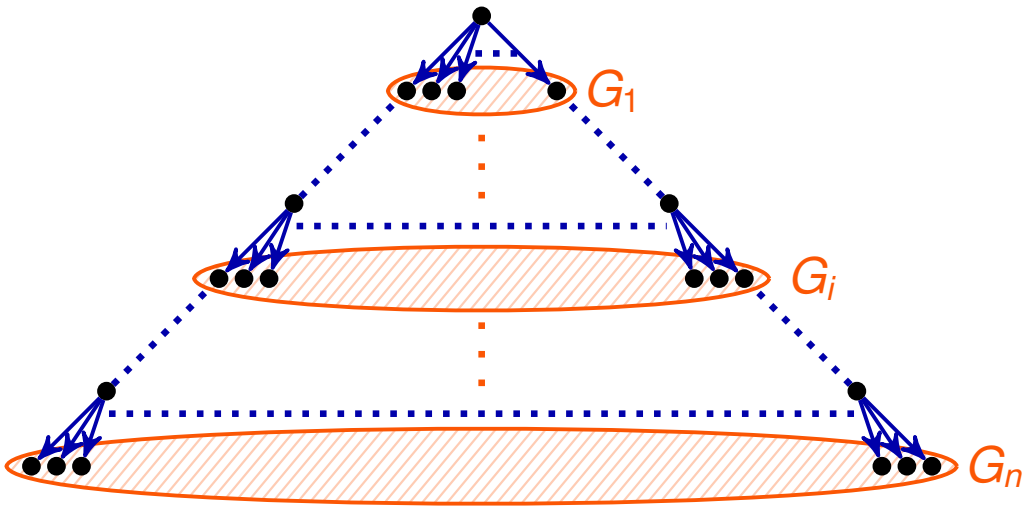

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PT on hyperfinite graphs

Theorem (Newman and Sohler, 2013 (Benjamini, Shramm, Shapira, Elek))
Every hyperfinite property is non-uniformly testable.

Hyperfinite graph classes include
- bounded tree-width graphs,
- graphs excluding a fixed minor.

Observation
The theorem does not say anything about the running time.
$\exists$ undecidable properties (of edgeless graphs) that are testable with constant query complexity by the theorem.

Typically: Any natural property that is testable with constant query complexity is also testable with constant running time.

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Bounded-degree model: Holy Grail II

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Characterise the properties that are uniformly testable with constant running time.

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Every MSO-definable property $\mathcal{P} \subseteq \mathcal{C}_d^t$ is uniformly testable with constant query complexity and polylogarithmic running time.

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Example: FO on degree ≤ 1

We consider graphs of degree ≤ 1.

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Let

$$\varphi = \neg\exists x_1 x_2 x_3 \Big(\bigwedge_{i \neq j} x_i \neq x_j \wedge \forall y (\neg E x_1 y) \wedge E x_2 x_3 \Big).$$

The only graphs satisfying φ are of the form

or

Let $G =$

Let $\varepsilon = 1/3$. Then G is ε-far from satisfying φ. However, w.h.p. we will not see this by sampling a constant number of vertices from G.

But φ is still testable: check if n is even or odd.

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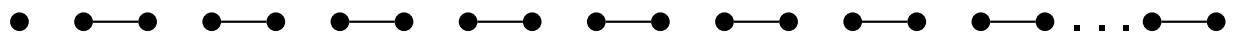
Let $G =$ 

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FO *on finitary graphs*

Let $b \in \mathbb{N}$. Class $\mathcal{C}$ is b-finitary, if every component of a graph in $\mathcal{C}$ has $\leq b$ vertices.

Theorem (A., Stimpson 2025+)

Let $\mathcal{C}$ be b-finitary. Then every FO*-definable property is testable on $\mathcal{C}$ with constant running time.*

Let $C_1, C_2, \ldots, C_t$ be an enumeration of all isomorphism types of components of size $\leq b$.

Definition

Every b-finitary graph G has a component histogram vector $\bar{v}_G$ where

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Capped component histograms

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For $k \in \mathbb{N}$, a k-capped component histogram vector is a vector $\bar{v}$ indexed by the C_i where every entry $\bar{v}[i]$ is either

- *a number $m \in \mathbb{N}$ with $0 \leq m < k$ ('rare components'), or*
- *'$\geq k$' ('frequent components').*

Example

Let $k = 10$, $\bar{v} = (0, 2, 0, \geq 10, 0, 0, 3, \geq 10, 0, 1, 0, 6, 0, \geq 10, \geq 10, 0, \ldots, 0)$.

Lemma (Corollary of Hanf's Theorem)

Let G be a b-finitary graph G and $\varphi \in$ FO. Checking whether $G \models \varphi$ amounts to checking whether $\bar{v}_G$ satisfies one of finitely many capped component histogram vectors.

Capped component histograms

Let $C_1, C_2, \ldots, C_t$ be an enumeration of all isomorphism types of components of size $\leq b$.

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FO *on finitary graphs*

Theorem (A., Stimpson 2025+)

Let $\mathcal{C}$ be b-finitary. Then every* FO*-definable property φ is testable on $\mathcal{C}$ with constant running time.

Proof.

Assume φ is given by a capped component histogram $\bar{v}$.
Let $n_0 := \sum_{C \text{ rare component}} \bar{v}[C] \cdot |C|$ and
$g := \gcd\{|C| : C \text{ frequent component}\}$.
Fix ε. Given oracle access to G and $n = |V(G)|$:

- Sample $s = s(\varepsilon)$ vertices from G and explore their b-neighbourhood & obtain good approximation $\bar{v}$ of G's b-neighbourhood distribution.
- If a rare component is found, reject.
- Otherwise, if $g|(n - n_0)$, accept, else reject.

Correctness: Assume n is large and $\bar{v}$ is good.
Show: If no rare comp/s found and $g|(n - n_0)$, then G can be turned into a graph satisfying φ by a small number of modifications ('Frobenius coin problem'; Newman-Sohler). □

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- Generalise approach to larger classes and logics
- Testing FO in the bounded-degree model: what is the query-complexity?

Theorem (Goldreich 2024)

Testing our property has query complexity $\Omega(\log\log\log\log n)$.

- Find the holy grails.

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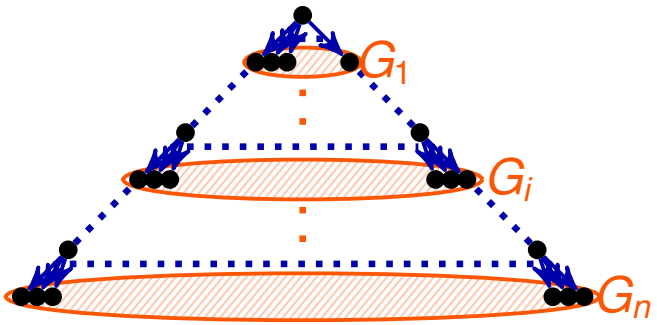

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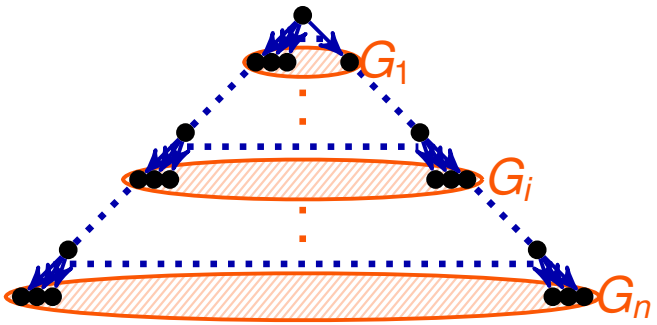

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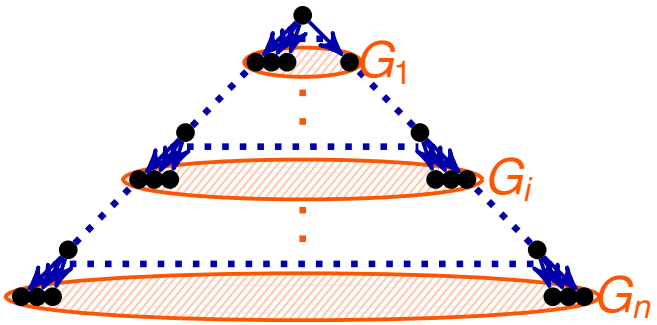

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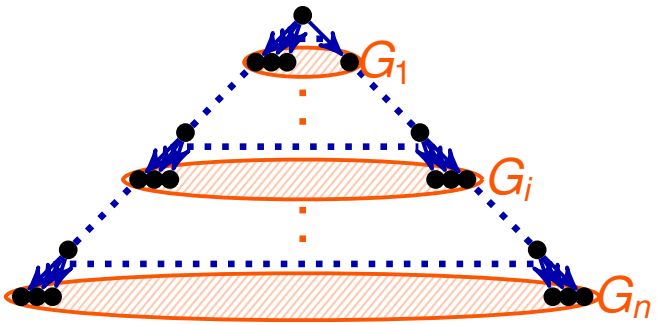

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Merci beaucoup!

Hyperfinite graphs

Let $0 \leq \varepsilon \leq 1$ and $k \in \mathbb{N}$.

- G is (ε, k)-hyperfinite if one can remove εn edges from G and obtain a graph whose connected components have size $\leq k$.
- Fix a function $\rho : \mathbb{R}^+ \to \mathbb{N}$.
  G is ρ-hyperfinite if G is $(\varepsilon, \rho(\varepsilon))$-hyperfinite for every $\varepsilon > 0$.
- A graph class $\mathcal{C}$ is hyperfinite if there is a function ρ such that every $G \in \mathcal{C}$ is ρ-hyperfinite.

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